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BY

HORATIO N. ROBINSON, LL.D.,

AUTHOR OF A FULL COURSE OF MATHEMATICS.

REWRITTEN BY

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PROFESSOR OF MATHEMATICS AND NATURAL PHILOSOPHY, UNIVERSITY OF ROCHESTER;
AUTHOR OF DIFFERENTIAL AND INTEGRAL CALCULUS.

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PREFACE.

In the preparation of this work, the author's previous treatise, Elements of Geometry, has formed the groundwork of construction. But in adapting the work to the present advanced state of Mathematical education in our best Institutions, it was found necessary so to alter the plan, and the arrangement of subjects, as to make this essentially a new work. The demonstrations of propositions have undergone radical changes, many new propositions have been introduced, and the number of Practical Problems greatly increased, so that the work is now believed to be as full and complete as could be desired in an elementary treatise.

In view of the fact that the Seventh Book is so much larger than the others, it may be asked why it is not divided into two. We answer, that classifications and divisions are based upon differences, and that the differences seized upon for this purpose must be determined by the nature of the properties and relations we wish to investigate. There is such a close resemblance between the geometrical properties of the polyedrons and the round bodies, and the demonstrations relating to the former require such slight modifications to become applicable to the latter, that there seems no sufficient reason for separating into two Books that part of Geometry which treats of them.

Practical rules with applications will be found throughout the work, and in addition to these, there is a full collection of carefully selected Practical Problems. These are given to exercise the powers and test the proficiency of the pupil, and when he has mastered the most or all of them, it is not likely that he will rest satisfied with present acquisition, but, conscious of augmented strength and certain of reward, he will enter new fields of investigation.

The author has been aided, in the preparation of the present work, by I. F. Quinby, A.M., of the University of Rochester, N. Y., late Professor of Mathematics in the United States Military Academy at West Point. The thorough scholarship and long and successful experience of this gentleman in the class-room, eminently qualify him for such a task; and to him the public are indebted for much that is valuable, both in the matter and arrangement of this treatise.

OCTOBER, 1860.

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GEOMETRY.

DEFINITIONS.

1. Geometry is the science which treats of position, and of the forms, measurements, mutual relations, and properties of limited portions of *space*.

SPACE extends without limit in all directions, and contains all bodies.

2. A **Point** is mere position, and has no magnitude.

3. Extension is a term employed to denote that property of bodies by virtue of which they occupy definite portions of space. The dimensions of extension are *length*, *breadth*, and *thickness*.

4. A **Line** is that which has extension in length only. The extremities of a line are points.

5. A **Right** or **Straight Line** is one all of whose parts lie in the same direction.

6. A **Curved Line** is one whose consecutive parts, however small, do not lie in the same direction.

7. A **Broken** or **Crooked Line** is composed of several straight lines, joined one to another successively, and extending in different directions.

When the word *line* is used, a straight line is to be understood, unless otherwise expressed.

8. A **Surface** or **Superficies** is that which has extension in length and breadth only.

9. A **Plane Surface**, or a **Plane**, is a surface such that if any two of its points be joined by a straight line, every point of this line will lie in the surface.

10. A **Curved Surface** is one which is neither a plane, nor composed of plane surfaces.

11. A **Plane Angle**, or simply an **Angle**, is the difference in the direction of two lines proceeding from the same point.

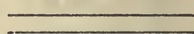
The other angles treated of in geometry will be named and defined in their proper connections.

12. A **Volume, Solid,** or **Body,** is that which has extension in length, breadth, and thickness.

These terms are used in a sense purely abstract, to denote mere space — whether occupied by matter or not, being a question with which geometry is not concerned.

Lines, Surfaces, Angles, and Volumes constitute the different kinds of quantity called *geometrical magnitudes.*

13. **Parallel Lines** are lines which have the same direction.

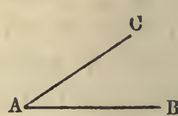


Hence parallel lines can never meet, however far they may be produced; for two lines taking the same direction cannot approach or recede from each other.

Two parallel lines cannot be drawn from the same point; for if parallel, they must coincide and form one line.

PLANE ANGLES.

To make an angle apparent, the two lines must meet in a point, as *AB* and *AC*, which meet in the point *A*.



Angles are measured by degrees.

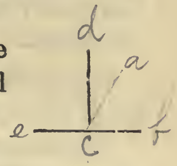
14. A **Degree** is one of the three hundred and sixty equal parts of the space about a point in a plane.

If, in the above figure, we suppose *AC* to coincide with *AB*, there will be but one line, and no angle; but if *AB* retain its position, and *AC* begin to revolve about the point *A*, an angle will be formed, and its magnitude will be expressed by that number of the

360 equal spaces about the point A, which is contained between AB and AC.

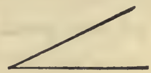
Angles are distinguished in respect to magnitude by the terms Right, Acute, and Obtuse Angles.

15. A **Right Angle** is that formed by one line meeting another, so as to make equal angles with that other.



The lines forming a right angle are *perpendicular to each other.*

16. An **Acute Angle** is less than a right angle.



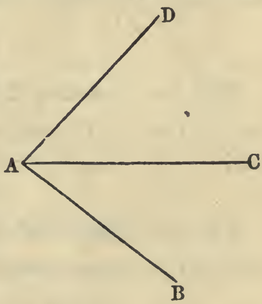
17. An **Obtuse Angle** is greater than a right angle.

Obtuse and acute angles are also called *oblique angles;* and lines which are neither parallel nor perpendicular to each other are called *oblique lines.*

18. The **Vertex** or **Apex** of an angle is the point in which the including lines meet.

19. An angle is commonly designated by a letter at its vertex; but when two or more angles have their vertices at the same point, they cannot be thus distinguished.

For example, when the three lines AB, AC, and AD meet in the common point A, we designate either of the angles formed, by three letters, placing that at the vertex between those at the opposite extremities of the including lines. Thus, we say, the angle BAC, etc.



20. Complements. — Two angles are said to be complements of each other, when their sum is equal to one right angle.

21. Supplements. — Two angles are said to be supplements of each other, when their sum is equal to two right angles.

PLANE FIGURES.

22. A **Plane Figure**, in geometry, is a portion of a plane bounded by straight or curved lines, or by both combined.

23. A **Polygon** is a plane figure bounded by straight lines, called the sides of the polygon.

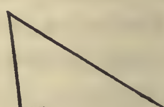
The least number of sides that can bound a polygon is three, and by the figure thus bounded all other polygons are analyzed.

FIGURES OF THREE SIDES.

24. A **Triangle** is a polygon having three sides and three angles.

Tri is a Latin prefix signifying three; hence a Triangle is literally a figure containing three angles. Triangles are denominated from the relations both of their sides and angles.

25. A **Scalene Triangle** is one in which no two sides are equal.



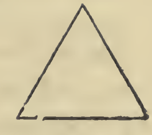
26. An **Isosceles Triangle** is one in which two of the sides are equal.

27. An **Equilateral Triangle** is one in which the three sides are equal.

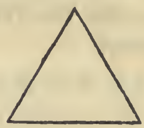
28. A **Right-Angled Triangle** is one which has one of the angles a right angle.

29. An **Obtuse-Angled Triangle** is one having an obtuse angle.

30. An **Acute-Angled Triangle** is one in which each angle is acute.



31. An **Equiangular Triangle** is one having its three angles equal.



Equiangular triangles are also equilateral, and vice versa.

FIGURES OF FOUR SIDES.

32. A **Quadrilateral** is a polygon having four sides and four angles.

33. A **Parallelogram** is a quadrilateral which has its opposite sides parallel.

Parallelograms are denominated from the relations both of their sides and angles.

34. A **Rectangle** is a parallelogram having its angles right angles.

35. A **Square** is an equilateral rectangle.

36. A **Rhomboid** is an oblique-angled parallelogram.

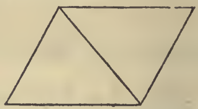
37. A **Rhombus** is an equilateral rhomboid.

38. A **Trapezium** is a quadrilateral having no two sides parallel.

39. A **Trapezoid** is a quadrilateral in which two opposite sides are parallel, and the other two oblique.

40. Polygons bounded by a greater number of sides than four are denominated only by the number of sides. A polygon of five sides is called a *Pentagon;* of six, a *Hexagon;* of seven, a *Heptagon;* of eight, an *Octagon;* of nine, a *Nonagon*, etc.

41. **Diagonals** of a polygon are lines joining the vertices of angles not adjacent.



42. The **Perimeter** of a polygon is its boundary considered as a whole.

43. The **Base** of a polygon is the side upon which the polygon is supposed to stand.

44. The **Altitude** of a polygon is the perpendicular distance between the base and a side or angle opposite the base.

45. **Equal Magnitudes** are those which are not only equal in all their parts, but which also, when applied the one to the other, will coincide throughout their whole extent.

46. **Equivalent Magnitudes** are those which, though they do not admit of coincidence when applied the one to the other, still have common measures, and are therefore numerically equal.

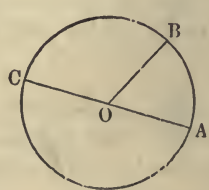
47. **Similar Figures** have equal angles, and the same number of sides.

Polygons may be similar without being equal; that is, the angles and the number of sides may be equal, and the *length* of the sides and the *size* of the figures unequal.

X

THE CIRCLE.

48. A **Circle** is a plane figure bounded by one uniformly curved line, all of the points in which are at the same distance from a certain point within, called the *Center*.



49. The **Circumference** of a circle is the curved line that bounds it.

50. The **Diameter** of a circle is a line passing through its center, and terminating at both ends in the circumference.

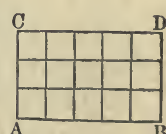
51. The **Radius** of a circle is a line extending from its center to any point in the circumference. It is one half of the diameter. All the diameters of a circle are equal, as are also all the radii.

52. An **Arc** of a circle is any portion of the circumference.

53. An angle having its vertex at the center of a circle is measured by the arc intercepted by its sides. Thus, the arc AB measures the angle AOB; and in general, to compare different angles, we have but to compare the arcs, included by their sides, of the equal circles having their centers at the vertices of the angles.

UNITS OF MEASURE.

54. The **Numerical Expression of a Magnitude** is a number expressing how many times it contains a magnitude of the same kind, and of known value, assumed as a unit. For lines, the measuring unit is any straight line of fixed value, as an inch, a foot, a rod, etc.; and for surfaces, the measuring unit is a square whose side may be any linear unit, as an inch, a foot, a mile, etc. The linear unit being arbitrary, the surface unit is equally so; and its selection is determined by considerations of convenience and propriety.



For example, the parallelogram $ABDC$ is measured by the number of *linear units* in CD, multiplied by the number of *linear units* in AC or BD; the product is the *square units* in $ABDC$. For, conceive CD to be composed of any number of equal parts—say five—and each part some unit of linear measure, and AC composed of three such units; from each point of division on CD draw lines parallel to AC, and from each point of division on AC draw lines parallel to CD or AB; then it is as obvious as an axiom that the parallelogram will contain $5 \times 3 = 15$ square units. Hence, to find the areas of right-angled parallelograms, *multiply the base by the altitude.*

EXPLANATION OF TERMS.

55. An **Axiom** is a self-evident truth, not only too simple to require, *but too simple to admit of, demonstration.*

56. A **Proposition** is something which is either proposed to be done, or to be demonstrated, and is either a problem or a theorem.

57. A **Problem** is something proposed to be done.

58. A **Theorem** is something proposed to be demonstrated.

59. A **Hypothesis** is a supposition made with a view to draw from it some consequence which establishes the truth or falsehood of a proposition, or solves a problem.

60. A **Lemma** is something which is premised, or demonstrated, in order to render what follows more easy.

61. A **Corollary** is a consequent truth derived immediately from some preceding truth or demonstration.

62. A **Scholium** is a remark or observation made upon something going before it.

63. A **Postulate** is a problem, the solution of which is self-evident.

POSTULATES.

Let it be granted—

I. That a straight line can be drawn from any one point to any other point;

II. That a straight line can be produced to any distance, or terminated at any point;

III. That the circumference of a circle can be described about any center, at any distance from that center.

AXIOMS.

1. *Things which are equal to the same thing are equal to each other.*

2. *When equals are added to equals the wholes are equal.*

3. *When equals are taken from equals the remainders are equal.*

4. *When equals are added to unequals the wholes are unequal.*

5. *When equals are taken from unequals the remainders are unequal.*

6. *Things which are double of the same thing, or equal things, are equal to each other.*

7. *Things which are halves of the same thing, or of equal things, are equal to each other.*

8. *The whole is greater than any of its parts.*

9. *Every whole is equal to all its parts taken together.*

10. *Things which coincide, or fill the same space, are identical, or mutually equal in all their parts.*

11. *All right angles are equal to one another.*

12. *A straight line is the shortest distance between two points.*

13. *Two straight lines cannot inclose a space.*

ABBREVIATIONS.

The common algebraic signs are used in this work, and demonstrations are sometimes made through the medium of equations; and it is so necessary that the student in geometry should understand some of the more simple operations of algebra, that we assume that he is acquainted with the use of the signs. As the terms circle, angle, triangle, hypothesis, axiom, theorem, corollary, and definition, are constantly occurring in a course of geometry, we shall abbreviate them as shown in the following list:

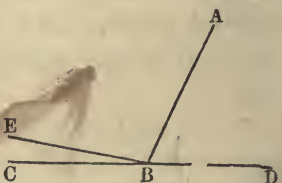
By Th. 1, any two supplementary angles, as *ABD*, *ABC*, are together equal to two right angles. And since the angular space about the point *B* is neither increased nor diminished by the number of lines drawn from that point, the sum of all the angles *DBA*, *ABE*, *EBH*, *HBC*, fills the same spaces as any two angles *HBD*, *HBC*. Hence the theorem; *from any point in a line, the sum of all the angles that can be formed on the same side of the line is equal to two right angles.*

Cor. 1. And, as the sum of all the angles that can be formed on the other side of the line, *CD*, is also equal to two right angles; therefore, *all the angles that can be formed quite round a point*, B, *by any number of lines, are together equal to four right angles.*

Cor. 2. Hence, also, the whole circumference of a circle, being the sum of the measures of all the angles that can be made about the center *F*, (Def. 53), is the measure of four right angles; consequently, a semicircumference, is the measure of two right angles; and a quadrant, or 90°, is the measure of one right angle.

THEOREM III.

If one straight line meets two other straight lines at a common point, forming two angles, which together are equal to two right angles the two straight lines are one and the same line.



Let the line *AB* meet the lines *BD* and *BE* at the common point *B*, making the sum of the two angles *ABD*, *ABE*, equal to two right angles; we are to prove that *DB* and *BE* are one straight line.

If DB and BE are not in the same line, produce DB to C, thus forming one line, DBC.

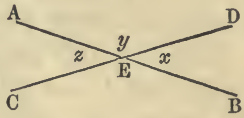
Now by Th. 1, $ABD + ABC$ must be equal to two right angles. But by hypothesis, $ABD + ABE$ is equal to two right angles.

Therefore, $ABD + ABC$ is equal to $ABD + ABE$, (Ax. 1). From each of these equals take away the common angle ABD, and the angle ABC will be equal to ABE, (Ax. 3). That is, the line BE must coincide with BC, and they will be in fact one and the same line, and they cannot be separated as is represented in the figure.

Hence the theorem; *if one line meets two other lines at a common point, forming two angles which together are equal to two right angles, the two lines are one and the same line.*

THEOREM IV.

If two straight lines intersect each other, the opposite or vertical angles must be equal.



If AB and CD intersect each other at E, we are to demonstrate that the angle AEC is equal to the vertical angle DEB; and the angle AED, to the vertical angle CEB.

As AB is one line met by DE, another line, the two angles AED and DEB, on the same side of AB, are equal to two right angles, (Th. 1). Also, because CD is a right line, and AE meets it, the two angles AEC and AED are together equal to two right angles.

Therefore, $AED + DEB = AEC + AED$. (Ax. 1.)

If from these equals we take away the common angle AED, the remaining angle DEB must be equal to the remaining angle AEC, (Ax. 3). In like manner, we can prove that AED is equal to CEB. Hence the theorem; *if the two lines intersect each other, the vertical angles must be equal.*

By Th. 1, any two supplementary angles, as ABD, ABC, are together equal to two right angles. And since the angular space about the point B is neither increased nor diminished by the number of lines drawn from that point, the sum of all the angles DBA, ABE, EBH, HBC, fills the same spaces as any two angles HBD, HBC. Hence the theorem; *from any point in a line, the sum of all the angles that can be formed on the same side of the line is equal to two right angles.*

Cor. 1. And, as the sum of all the angles that can be formed on the other side of the line, CD, is also equal to two right angles; therefore, *all the angles that can be formed quite round a point,* B, *by any number of lines, are together equal to four right angles.*

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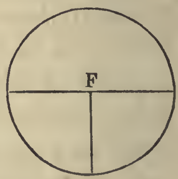
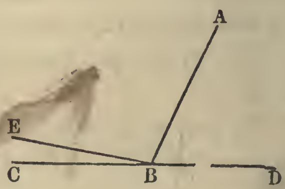


THEOREM III.

If one straight line meets two other straight lines at a common point, forming two angles, which together are equal to two right angles the two straight lines are one and the same line.

Let the line AB meet the lines BD and BE at the common point B, making the sum of the two angles ABD, ABE, equal to two right angles; we are to prove that DB and BE are one straight line.



If DB and BE are not in the same line, produce DB to C, thus forming one line, DBC.

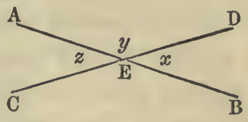
Now by Th. 1, $ABD + ABC$ must be equal to two right angles. But by hypothesis, $ABD + ABE$ is equal to two right angles.

Therefore, $ABD + ABC$ is equal to $ABD + ABE$, (Ax. 1). From each of these equals take away the common angle ABD, and the angle ABC will be equal to ABE, (Ax. 3). That is, the line BE must coincide with BC, and they will be in fact one and the same line, and they cannot be separated as is represented in the figure.

Hence the theorem; *if one line meets two other lines at a common point, forming two angles which together are equal to two right angles, the two lines are one and the same line.*

THEOREM IV.

If two straight lines intersect each other, the opposite or vertical angles must be equal.



If AB and CD intersect each other at E, we are to demonstrate that the angle AEC is equal to the vertical angle DEB; and the angle AED, to the vertical angle CEB.

As AB is one line met by DE, another line, the two angles AED and DEB, on the same side of AB, are equal to two right angles, (Th. 1). Also, because CD is a right line, and AE meets it, the two angles AEC and AED are together equal to two right angles.

Therefore, $AED + DEB = AEC + AED$. (Ax. 1.)

If from these equals we take away the common angle AED, the remaining angle DEB must be equal to the remaining angle AEC, (Ax. 3). In like manner, we can prove that AED is equal to CEB. Hence the theorem; *if the two lines intersect each other, the vertical angles must be equal.*

Second Demonstration.

By Def. 11, the angle DEB is the difference in the direction of the lines ED and EB; and the angle AEC is the difference in the direction of the lines EC and EA.

But ED is opposite in direction to EC; and EB is opposite in direction to EA.

Hence, the difference in the direction of ED and EB is the same as that of EC and EA, as is obvious by inspection.

Therefore, the angle DEB is equal to its opposite AEC.

In like manner, we may prove $AED = CEB$.

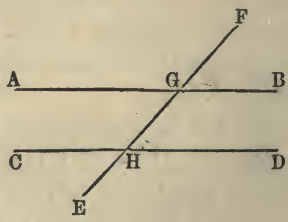
Hence the theorem; *if two lines intersect each other, the vertical angles must be equal.*

THEOREM V.

If a straight line intersects two parallel lines, the sum of the two interior angles on the same side of the intersecting line is equal to two right angles.

[Note.—By *interior* angles, we mean angles which lie between the parallels; the *exterior* angles are those not between the parallels.]

Let the line EF intersect the parallels AB and CD; then we are to demonstrate that the angles $BGH + GHD = 2$ R. ⌊



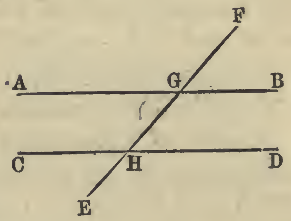
Because GB and HD are parallel, they are equally inclined to the line EF, or have the same difference of direction from that line. Therefore, ⌊ $FGB =$ ⌊ GHD. To each of these equals add the ⌊ BGH, and we have $FGB + BGH = GHD + BGH$.

But by Th. 1, the first member of this equation is equal to two right angles; and the second member is the sum of the two angles between the parallels. Hence the theorem; *if a line intersects two parallel lines, the sum of the two interior angles on the same side of the intersecting line must be equal to two right angles.*

SCHOLIUM.—As AB and CD are parallel lines, and EF is a line intersecting them, AB and EF must make angles equal to those made by CD and EF. That is, the angles about the point G must be equal to the corresponding angles about the point H.

THEOREM VI.

If a line intersects two parallel lines, the alternate interior angles are equal.



Let AB and CD be parallels, intersected by EF at H and G. Then we are to prove that the angle AGH is equal to the alternate angle GHD, and $CHG = HGB$.

By Th. 5, $\angle BGH + \angle GHD =$ two right angles. Also, by Th. 1, $\angle AGH + \angle BGH =$ two right angles. From these equals take away the common angle BGH, and $\angle GHD$ will be left, equal to $\angle AGH$, (Ax. 3). In like manner, we can prove that the angle CHG is equal to the angle HGB. Hence the theorem; *if a line intersects two parallel lines, the alternate interior angles are equal.*

Cor. 1. Since $\angle AGH = \angle FGB$,
and $\angle AGH = \angle GHD$;
Therefore, $\angle FGB = \angle GHD$ (Ax. 1).
Also, $\angle AGF + \angle AGH = 2$ R. $\angle$, (Th. 1),
and $\angle CHG + \angle AGH = 2$ R. $\angle$, (Th. 5);
Therefore,
$\angle AGF + \angle AGH = \angle CHG + \angle AGH$, (Ax. 1);
and $\angle AGF = \angle CHG$, (Ax. 3).

That is, *the exterior angle is equal to the interior opposite angle on the same side of the intersecting line.*

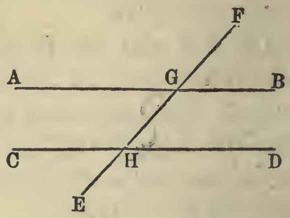
Cor. 2. Since $\angle AGH = \angle FGB$,
and $\angle AGH = \angle CHE$;
Therefore, $\angle FGB = \angle CHE$.
In the same manner it may be shown that

$$\angle AGF = \angle EHD.$$

Hence, *the alternate exterior angles are equal.*

THEOREM VII.

If a line intersects two other lines, making the sum of the two interior angles on the same side of the intersecting line equal to two right angles, the two straight lines are parallel.



Let the line EF intersect the lines AB and CD, making the two angles $BGH + GHD$ = to two right angles; then we are to demonstrate that AB and CD are parallel.

As EF is a right line and BG meets it, the two angles FGB and BGH are together equal to two right angles, (Th. 1). But by hypothesis, the angles, BGH and GHD, are together equal to two right angles. From these two equals take away the common angle BGH, and the remaining angles FGB and GHD must be equal, (Ax. 3). Now, because GB and HD make equal angles with the same line EF, they must extend in the same direction; and lines having the same direction are parallel, (Def. 13). Hence the theorem; *if a line intersects two other lines, making the sum of the two interior angles on the same side of the intersecting line equal to two right angles, the two lines must be parallel.*

Cor. 1. If a line intersects two other lines, making the alternate interior angles equal, the two lines intersected must be parallel.

Suppose the $\llcorner AGH = \llcorner GHD$. Adding $\llcorner HGB$ to each, we have

$$\llcorner AGH + \llcorner HGB = \llcorner GHD + \llcorner HGB.$$

but the first member of this equation, that is, $\llcorner AGH + \llcorner HGB$, is equal to two right angles; hence the second member is also equal to the same; and by the theorem, the lines AB and CD are parallel.

Cor. 2. If a line intersects two other lines, making the opposite exterior and interior angles equal, the two lines intersected must be parallel.

Suppose the $\angle FGB = \angle GHD$. Adding the $\angle HGB$ to each, we have

$$\angle FGB + \angle HGB = \angle GHD + HGB.$$

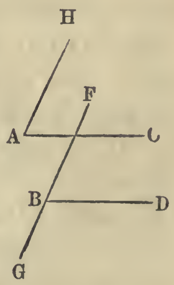
But the first member of this equation is equal to two right angles; hence the second member is also equal to two right angles; and by the theorem, the lines AB and CD are parallel.

Cor. 3. If a line intersects two other lines, making the alternate exterior angles equal, the lines must be parallel.

Suppose $\angle BGF = \angle CHE$, and $\angle AGF = \angle DHE$.
By Th. 4, $\angle BGF = \angle AGH$, and $\angle CHE = \angle DHG$.
And since $\angle BGF = \angle CHE$, $\angle AGH = \angle DHG$.
That is, the alternate interior angles are equal; and hence (by Cor. 1) the two lines are parallel.

THEOREM VIII.

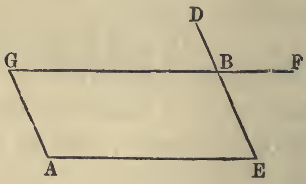
If two angles have their sides parallel, the two angles will be either equal or supplementary.



Let AC be parallel to BD, and AH parallel to BF or to BG. Then we are to prove that the angle DBF is equal to the angle CAH, and that the angle DBG is supplementary to the angle A. The angle CAH is formed by the difference in the direction of AC and AH; and the angle DBF is formed by the difference in the direction of BD and BF. But AC and AH have the same directions as BD and BF, because they are respectively parallel. Therefore, by Def. 11, $\angle CAH = \angle DBF$. But the line BG has the same direction as BF, and the angle DBG is supplementary to DBF. Hence the theorem; *angles whose sides are parallel are either equal or supplementary.*

THEOREM IX.

The opposite angles of any parallelogram are equal.



Let $AEBG$ be a parallelogram. Then we are to prove that the angle GBE is equal to its opposite angle A.

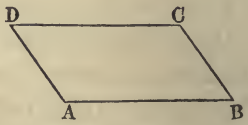
Produce EB to D, and GB to F; then, since BD is parallel to AG, and BF to AE, the angle DBF is equal to the angle A, (Th. 8).

But the angles GBE and DBF, being vertical, are equal, (Th. 4). Therefore, the opposite angles GBE and A, of the parallelogram $AEBG$, are equal.

In like manner, we can prove the angle E equal to the angle G. Hence the theorem; *the opposite angles of any parallelogram are equal.*

THEOREM X.

The sum of the angles of any parallelogram is equal to four right angles.

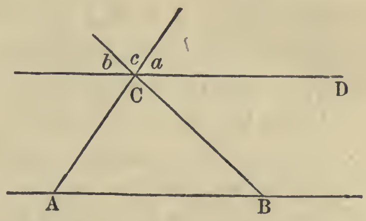


Let $ABCD$ be a parallelogram. We are to prove that the sum of the angles A, B, C and D, is equal to four right angles, or to 360°.

Because AD and BC are parallel lines, and AB intersects them, the two interior angles A and B are together equal to two right angles, (Th. 5). And because CD intersects the same parallels, the two interior angles C and D are also together equal to two right angles. By addition, we have the sum of the four interior angles of the parallelogram $ABCD$, equal to four right angles. Hence the theorem; *the sum of the angles of any parallelogram is equal to four right angles.*

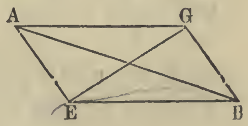
THEOREM XI.

The sum of the three angles of any triangle is equal to two right angles.



Let ABC be a triangle, and through its vertex C draw a line parallel to the base AB, and produce the sides AC and BC. Then the angles A and a, being exterior and interior opposite angles on the same side of the line AC, are equal to each other. For the same reason, $\llcorner B = \llcorner b$. The angles C and c, being vertical angles, are also equal, (Th. 4). Therefore, the angles A, B, C are equal to the angles a, b, c respectively. But the angles around the point C, on the upper side of the parallel CD, are equal to two right angles, (by Th. 2). Hence the theorem; *the sum of the three angles, etc.*

Second Demonstration.



Let $AEBG$ be a parallelogram. Draw the diagonal GE; thus dividing the parallelogram into two triangles, and the opposite angles G and E each into two angles.

Because GB and AE are parallel, the *alternate interior* angles BGE and GEA are equal, (Th. 6). Designate each of these by b.

In like manner, because EB and AG are parallel, the alternate interior angles, BEG and EGA, are equal. Designate each of these by a.

Now we are to prove that the three angles B, b, and a, and also that the three angles A, a, and b, are equal to two right angles.

Because A and B are opposite angles of a parallelogram, they are equal, (Th. 9), and $\angle A + \angle B = 2 \angle A$.

And all the interior angles of the parallelogram are equal to four right angles, (Th. 10).

Therefore, $2A + 2a + 2b = 4$ right angles.

Dividing by 2, and $A + a + b = 2$ "

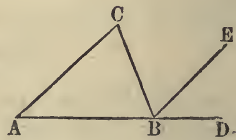
That is, all the angles of the triangle AGE are together equal to two right angles

Hence the theorem; *the sum of the three angles, etc.*

SCHOLIUM.—Any triangle, as AGE, may be conceived to be part of a parallelogram. For, let AGE be drawn independently of the parallelogram; then draw EB from the point E parallel to AG, and through the point G draw GB parallel to AE, and a parallelogram will be formed embracing the triangle; and thus the sum of the three angles of any triangle is proved equal to two right angles.

This truth is so fundamental, important, and practical, as to require special attention; we therefore give a

Third Demonstration.



Let ABC be a triangle. Then we are to show that the angles A, C, and ABC, are together equal to two right angles.

Let AB be produced to D, and from B draw BE parallel to AC.

Then, EBD and CAB being exterior and interior opposite angles on the same side of the line AD, are equal, (Th. 6, Cor. 1). Also, CBE and ACB, being alternate angles, are equal, (Th. 6).

By addition, observing that $\angle CBE$, added to $\angle EBD$, must make $\angle CBD$, we have

$$\angle CBD = \angle A + \angle C. \qquad (1.)$$

To each of these equals add the angle CBA, and we shall have

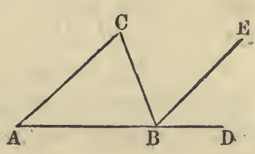
$$\angle CBA + \angle CBD = \angle A + \angle C + \angle CBA.$$

But (by Th. 1), the sum of the first two is equal to two right angles; therefore, the three angles, A, C, and CBA, are together equal to two right angles.

Hence the theorem; *the sum of the three angles, etc.*

THEOREM XII.

If any side of a triangle is produced, the exterior angle is equal to the sum of the two interior opposite angles.



Let ABC be a triangle. Produce AB to D; and we are to prove that the angle CBD is equal to the sum of the two angles A and C.

We establish this theorem by a course of reasoning in all respects the same as that by which we obtained Eq. (1.), third demonstration, (Th. 11).

Cor. 1. Since the exterior angle of any triangle is equal to the sum of the two interior opposite angles, therefore it is greater than either one of them.

Cor. 2. If two angles in one triangle be equal to two angles in another triangle, the third angles will also be equal, each to each, (Ax. 3); that is, the two triangles will be mutually equiangular.

Cor. 3. If one angle in a triangle be equal to one angle in another, the sum of the remaining angles in the one will also be equal to the sum of the remaining angles in the other, (Ax. 3).

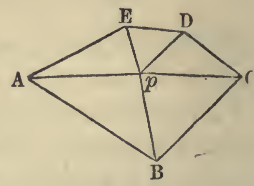
Cor. 4. If one angle of a triangle be a right angle, the sum of the other two will be equal to a right angle, and each of them singly will be acute, or less than a right angle.

Cor. 5. The two smaller angles of every triangle are acute, or each is less than a right angle.

Cor. 6. All the angles of a triangle may be acute, but no triangle can have more than one right or one obtuse angle.

THEOREM XIII.

In any polygon, the sum of all the interior angles is equal to twice as many right angles, less four, as the figure has sides.



Let $ABCDE$ be any polygon; we are to prove that the sum of all its interior angles, $A+B+C+D+E$, is equal to twice as many right angles, less four, as the figure has sides.

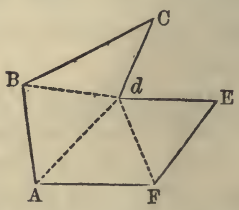
From any point, p, within the figure, draw lines pA, pB, pC, etc., to all the angles, thus dividing the polygon into as many triangles as it has sides. Now, the sum of the three angles of each of these triangles is equal to two right angles, (Th. 11); and the sum of the angles of all the triangles must be equal to twice as many right angles as the figure has sides. But the sum of these angles contains the sum of four right angles about the point p; taking these away, and the remainder is the sum of the interior angles of the figure. Therefore, the sum must be equal to twice as many right angles, less four, as the figure has sides.

Hence the theorem; *in any polygon, etc.*

From this Theorem is derived the rule for finding the sum of the interior angles of any right-lined figure:

Subtract 2 *from the number of sides, and multiply the remainder by* 2; *the product will be the number of right angles.*

Thus, if the number of sides be represented by S, the number of right angles will be represented by $(2S-4)$.



The Theorem is not varied in case of a re-entrant angle, as represented at d, in the figure $ABCdEF$.

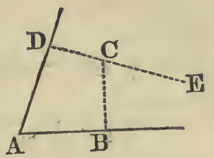
Draw lines from the angle d to the several opposite angles, making as many triangles as the figure has sides, *less two*, and the sum of the three angles of each triangle equals two right angles.

THEOREM XIV.

If the sides of one angle be respectively perpendicular to the sides of a second angle, these two angles will be either equal or supplementary.

Again

Let BAD be the first angle, and from any point within it, as C, draw CB and CD, at right angles, the first to AB, and the second to AD, and produce CD in the direction CE, thus forming at C the supplementary angles BCE, BCD; then will the angle BCE be equal to the angle A, and therefore BCD, which is the supplement of BCE, will also be the supplement of the angle A.



For since $ABCD$ is a quadrilateral, the sum of the four interior angles is four right angles (Prop. 13), and because the angles ABC and ADC are each right angles, the sum of the angles BAD, BCD is two right angles. But the sum of the adjacent angles BCE, BCD is also two right angles. Hence, if in these last two sums we omit the common angle BCD, we have remaining the angle BCE, equal to the angle BAD, and consequently the angle BCD which is the supplement of the first of these equal angles is also the supplement of the other.

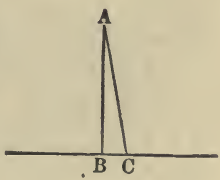
Hence the Theorem.

Scholium.—If the vertex of the second angle be without the first angle, we would draw through any assumed point within the first angle parallels to the sides of the second; the above demonstration will then apply to the first angle, and the angle formed by the parallels.

THEOREM XV.

From any point without a straight line, but one perpendicular can be drawn to that line.

From the point A let us suppose it possible that two perpendiculars, AB and AC, can be drawn. Now, because AB is a supposed perpendicular, the angle ABC is a right angle; and because AC is a supposed per-



pendicular, the angle ACB is also a right angle; and if two angles of the triangle ABC are together equal to two right angles, the third angle, BAC, must be infinitely small, or zero; that is, the two perpendiculars being drawn through the common point A, and including no angle, must necessarily coincide, and form one and the same perpendicular.

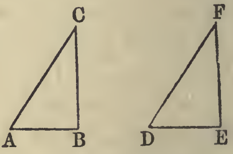
Hence the theorem; *from any point without a straight line, etc.*

Cor. At a given point in a straight line but one perpendicular can be erected to that line; for, if there could be two perpendiculars, we should have unequal righ angles, which is impossible.

THEOREM XVI.

Two triangles which have two sides and the included angle in the one, equal to two sides and the included angle in the other, each to each, are equal in all respects.

In the two $\triangle$'s, ABC and DEF, on the supposition that $AB = DE$, $AC = DF$, and $\angle A = \angle D$, we are to prove that BC must $= EF$, the $\angle B = \angle E$, and the $\angle C = \angle F$.



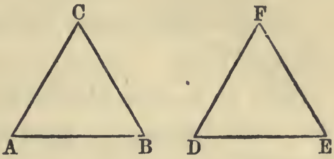
Conceive the $\triangle ABC$ cut out of the paper, taken up, and placed on the $\triangle DEF$ in such a manner that the point A shall fall on the point D, and the line AB on the line DE; then the point B will fall on the point E, because the lines are equal. Now, as the $\angle A = \angle D$, the line AC must take the same direction as DF, and fall on DF; and as $AC = DF$, the point C will fall on F. B being on E and C on F, BC must be exactly on EF, (otherwise, two straight lines would enclose a space, Ax. 13), and $BC = EF$, and the two magnitudes exactly fill the same space. Therefore, $BC = EF$, $\angle B = \angle E$, $\angle C = \angle F$, and the two $\triangle$'s are equal, (Ax. 10).

Hence the theorem; *two triangles which have two sides, etc.*

THEOREM XVII.

When two triangles have a side and two adjacent angles in the one, equal to a side and two adjacent angles in the other, each to each, the two triangles are equal in all respects.

In two $\triangle$'s, as ABC and DEF, on the supposition that $BC = EF$, $\angle B = \angle E$, and $\angle C = \angle F$, we are to prove that $AB = DE$, $AC = DF$, and $\angle A = \angle D$.



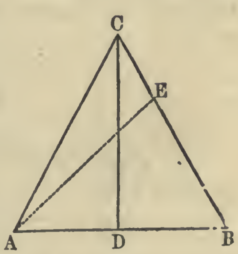
Conceive the $\triangle$ ABC taken up and placed on the $\triangle$ DEF, so that the side BC shall exactly coincide with its equal side EF; now, because the angle B is equal to the angle E, the line BA will take the direction of ED, and will fall exactly upon it; and because the angle C is equal to the angle F, the line CA will take the direction of FD, and fall exactly upon it; and the two lines BA and CA, exactly coinciding with the two lines ED and FD, the point A will fall on D, and the two magnitudes will exactly fill the same space; therefore, by Ax. 10, they are equal, and $AB = DE$, $AC = DF$, and the $\angle A = \angle D$.

Hence the theorem; *when two triangles have a side and two adjacent angles in the one, equal to, etc.*

THEOREM XVIII.

If two sides of a triangle are equal, the angles opposite to these sides are also equal.

Let ABC be a triangle; and on the supposition that $AC = BC$, we are to prove that the $\angle A$ = the $\angle B$.



Conceive the angle C divided into two equal angles by the line CD; then we have two $\triangle$'s, ADC and BDC, which have the two sides, AC and CD of the one, equal to the two sides, CB and CD of the other; and the included angle ACD, of the one, equal to the included angle BCD of the other: therefore, (Th. 16), $AD = BD$, and the angle A, opposite to CD of the one triangle, is equal to the angle B, opposite to CD of the other triangle; that is, $\angle A = \angle B$.

Hence the theorem; *if two sides of a triangle are equal, the angles, etc.*

Cor. 1. Conversely: *if two angles of a triangle are equal, the sides opposite to them are equal, and the triangle is isosceles.*

For, if AC is not equal to BC, suppose BC to be the greater, and make $BE = AE$; then will $\triangle\ AEB$ be isosceles, and $\angle EAB = \angle EBA$; hence $\angle EAB = \angle CAB$, or a part is equal to the whole, which is absurd; therefore, CB cannot be greater than AC, that is, neither of the sides AC, BC, can be greater than the other, and consequently they are equal.

Cor. 2. As the two triangles, ACD and BCD, are in all respects equal, the line which bisects the angle included between the equal sides of an isosceles $\triangle$ also bisects the base, and is perpendicular to the base.

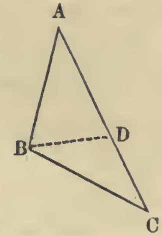
Scholium 1.—If in the perpendicular DC, any other point than C be taken, and lines be drawn to the extremities A and B, such lines will be equal, as is evident from Th. 16; hence, we may announce this truth: *Any point in a perpendicular drawn from the middle of a line, is at equal distances from the two extremities of the line.*

Scholium 2.—Since two points determine the position of a line, it follows, *that a line which joins two points each equally distant from the extremities of a given line, is perpendicular to this line at its middle point.*

THEOREM XIX.

The greater side of every triangle has the greater angle opposite to it.

Let ABC be a $\triangle$; and on the supposition that AC is greater than AB, we are to prove that the angle ABC is greater than the $\angle C$. From AC, the greater of the two sides, take AD, equal to the less side AB, and draw BD, thus making two triangles of the original triangle. As $AB = AD$, the $\angle ADB =$ the $\angle ABD$, (Th. 18).



But the $\angle ADB$ is the exterior angle of the $\triangle BDC$, and is therefore greater than C, (Th. 12, Cor. 1); that is, the $\angle ABD$ is greater than the $\angle C$. Much more, then, is the angle ABC greater than the angle C.

Hence the theorem; *the greater side of every triangle, etc.*

Cor. Conversely: *the greater angle of any triangle has the greater side opposite to it.*

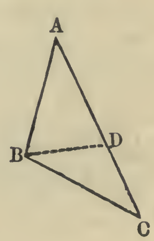
In the triangle ABC, let the angle B be greater than the angle A; then is the side AC greater than the side BC.

For, if $BC = AC$, the angle A must be equal to the angle B, (Th. 18), which is contrary to the hypothesis; and if $BC > AC$, the angle A must be greater than the angle B, by what is above proved, which is also contrary to the hypothesis; hence BC can be neither equal to, nor greater, than AC; it is therefore less than AC.

THEOREM XX.

The difference between any two sides of a triangle is less than the third side.

Let ABC be a $\triangle$, in which AC is greater than AB; then we are to prove that $AC - AB$ is less than BC.



On AC, the greater of the two sides, lay off AD equal to AB.

Now, as a straight line is the shortest distance between two points, we have

$$AB + BC > AC. \qquad (1)$$

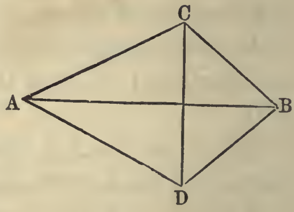
From these unequals subtract the equals $AB = AD$, and we have $BC > AC - AB$. (Ax. 5).

Hence the theorem; *the difference between any two sides of a triangle, etc.*

THEOREM XXI.

If two triangles have the three sides of the one equal to the three sides of the other, each to each, the two triangles are equal, and the equal angles are opposite the equal sides.

In two triangles, as ABC and ABD, on the supposition that the side AB of the one = the side AB of the other, $AC = AD$, and $BC = BD$, we are to demonstrate that $\angle ACB = \angle ADB$, $\angle BAC = \angle BAD$, and $\angle ABC = \angle ABD$.



Conceive the two triangles to be joined together by their longest equal sides, and draw the line CD.

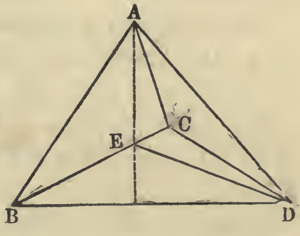
Then, in the triangle ACD, because AC is equal to AD, the angle ACD is equal to the angle ADC, (Th. 18). In like manner, in the triangle BCD, because BC is equal to BD, the angle BCD is equal to the angle BDC. Now, the angle ACD being equal to the angle ADC, and the angle BCD to the angle BDC, $\angle ACD + \angle BCD = \angle ADC + \angle BDC$, (Ax. 2); that is, the whole angle ACB is equal to the whole angle ADB.

Since the two sides AC and CB are equal to the two sides AD and DB, each to each, and their included angles ACB, ADB, are also equal, the two triangles ABC, ABD, are equal, (Th. 16), and have their other angles equal; that is, $\angle BAC = \angle BAD$, and $\angle ABC = \angle ABD$.

Hence the theorem; *if two triangles have the three sides of the one, etc.*

THEOREM XXII.

If two triangles have two sides of the one equal to two sides of the other, each to each, and the included angles unequal, the third sides will be unequal, and the greater third side will belong to the triangle which has the greater included angle.



In the two $\triangle$'s, ABC and ACD, let AB and AC of the one $\triangle$ be equal to AD and AC of the other $\triangle$, and the angle BAC greater than the angle DAC; we are to prove that the side BC is greater than the side CD.

Conceive the two $\triangle$'s joined together by their shorter equal sides, and draw the line BD. Now, as $AB = AD$, ABD is an isosceles $\triangle$. From the vertex A, draw a line bisecting the angle BAD. This line must be perpendicular to the base BD, (Th. 18, Cor. 2). Since the $\angle BAC$ is greater than the $\angle DAC$, this line must meet BC, and will not meet CD. From the point E, where the perpendicular meets BC, draw ED.

Now $BE = DE$, (Th. 18, Scholium 1).

Add EC to each; then $BC = DE + EC$.

But $DE + EC$ is greater than DC.

Therefore $BC > DC$.

Hence the theorem; *if two triangles have two sides of one equal to two sides of the other, etc.*

Cor. Any point out of the perpendicular drawn from the middle point of a line, is unequally distant from the extremities of the line.

4

X

X

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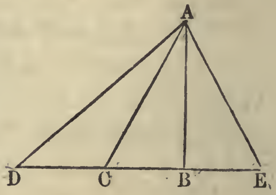
THEOREM XXIII.

A perpendicular is the shortest line that can be drawn from any point to a straight line; and if other lines be drawn from the same point to the same straight line, that which meets it farthest from the perpendicular will be longest; and lines at equal distances from the perpendicular, on opposite sides, are equal.



Let A be any point without the line DE; let AB be the perpendicular; and AC, AD, and AE oblique lines: then, if BC is less than BD, and $BC = BE$, we are to show,

1st. That AB is less than AC.

2d. That AC is less than AD. 3d. That $AC = AE$.

1st. In the triangle ABC, as AB is perpendicular to BC, the angle ABC is a right angle; and, therefore (by Theorem 12, Cor. 4); the angle BCA is less than a right angle; and, as the greater side is always opposite the greater angle, AB is less than AC; and AC may be any line not identical with AB; therefore a perpendicular is the shortest line that can be drawn from A to the line DE.

2d. As the two angles, ACB and ACD, are together equal to two right angles, (Th. 1), and ACB is less than a right angle, ACD must be greater than a right angle; consequently, the $\angle D$ is less than a right angle; and, in the $\triangle$ ACD, AD is greater than AC, or AC is less than AD, (Th. 19 Cor).

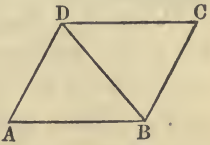
3d. In the $\triangle$'s ABC and ABE, AB is common, $CB = BE$, and the angles at B are right angles; therefore, $AC = AE$, (Th. 16).

Hence the theorem; *a perpendicular is the shortest line etc.*

Cor. Conversely: if two equal oblique lines be drawn from the same point to a given straight line, they will meet the line at equal distances from the foot of the perpendicular drawn from that point to the given line.

THEOREM XXIV.

The opposite sides, and also the opposite angles of any parallelogram, are equal.



Let $ABCD$ be a parallelogram. Then we are to show that $AB = DC$, $AD = BC$, $\angle A = \angle C$, and $\angle ADC = \angle ABC$.

Draw a diagonal, as BD; now, because AB and DC are parallel, the alternate angles ABD and BDC are equal, (Th. 6). For the same reason, as AD and BC are parallel, the angles ADB and DBC are equal. Now, in the two triangles ABD and BCD, the side BD is common,

$$\text{the } \angle ADB = \angle DBC \qquad (1)$$
$$\text{and } \angle BDC = \angle ABD \qquad (2)$$

Therefore, the angle $A =$ the angle C, and the two triangles are equal in all respects, (Th. 17); that is, the sides opposite the equal angles are equal; or, $AB = DC$, and $AD = BC$. By adding equations (1) and (2), we have the angle $ADC =$ the angle ABC, (Ax. 2).

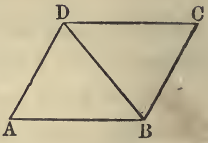
Hence the theorem; *the opposite sides, and the opposite angles, etc.*

Cor. 1. As the sum of all the angles of a parallelogram is equal to four right angles, and the angle A is always equal to the opposite angle C; therefore, if A is a right angle, C is also a right angle, and the figure is a rectangle.

Cor. 2. As the angle ABC, added to the angle A, gives the same sum as the angles of the $\triangle ADB$; therefore, the two adjacent angles of a parallelogram are together equal to two right angles.

THEOREM XXV.

If the opposite sides of a quadrilateral are equal, they are also parallel, and the figure is a parallelogram.



Let $ABCD$ be any quadrilateral; on the supposition that $AD = BC$, and $AB = DC$, we are to prove that AD is parallel to BC, and AB parallel to DC.

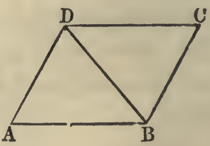
Draw the diagonal BD; we now have two triangles, ABD and BCD, which have the side BD common, AD of the one $= BC$ of the other, and AB of the one $= CD$ of the other; therefore the two $\triangle$'s are equal, (Th. 21), and the angles opposite the equal sides are equal; that is, the angle $ADB =$ the angle CBD; but these are alternate angles; hence, AD is parallel to BC, (Th. 7, Cor. 1); and because the angle $ABD =$ the angle BDC, AB is parallel to CD, and the figure is a parallelogram.

Hence the theorem; *if the opposite sides of a quadrilateral, etc.*

Cor. This theorem, and also Th. 24, proves that the two $\triangle$'s which make up the parallelogram are equal; and the same would be true if we drew the diagonal from A to C; *therefore, the diagonal of any parallelogram bisects the parallelogram.*

THEOREM XXVI.

The lines which join the corresponding extremities of two equal and parallel straight lines, are themselves equal and parallel; and the figure thus formed is a parallelogram.



On the supposition that AB is equal and parallel to DC, we are to prove that AD is equal and parallel to BC; and that the figure is a parallelogram.

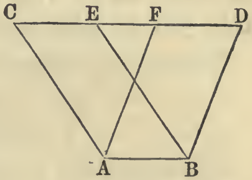
Draw the diagonal BD; now, since

AB and DC are parallel, and BD joins them, the alternate angles ABD and BDC are equal; and since the side $AB =$ the side DC, and the side BD is common to the two △'s ABD and CDB, therefore the two triangles are equal, (Th. 16); that is, $AD = BC$, the angle $A = C$, and the ∟ $ADB =$ the ∟ DBC; also AD is parallel to BC; and the figure is a parallelogram.

Hence the theorem; *the lines which join the corresponding extremities, etc.*

THEOREM XXVII.

Parallelograms on the same base, and between the same parallels, are equivalent, or equal in respect to area or surface.



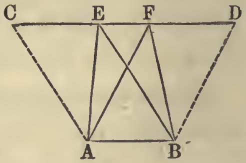
Let $ABEC$ and $ABDF$ be two parallelograms on the same base AB, and between the same parallels AB and CD; we are to prove that these two parallelograms are equal.

Now, CE and FD are equal, because they are each equal to AB, (Th. 24); and, if from the whole line CD we take, in succession, CE and FD, there will remain $ED = CF$, (Ax. 3); but $BE = AC$, and $AF = BD$, (Th. 24); hence we have two △'s, CAF and EBD, which have the three sides of the one equal to the three sides of the other, each to each; therefore, the two △'s are equal, (Th. 21). If, from the whole figure $ABDC$, we take away the △ CAF, the parallelogram $ABDF$ will remain; and if from the whole figure we take away the other △ EBD, the parallelogram $ABEC$ will remain. Therefore, (Ax. 3), the parallelogram $ABDF =$ the parallelogram $ABEC$.

Hence the theorem; *Parallelograms on the same base, etc.*

THEOREM XXVIII.

Triangles on the same base and between the same parallels are equivalent.



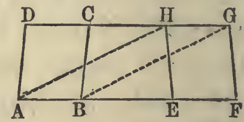
Let the two $\triangle$'s ABE and ABF have the same base AB, and be between the same parallels AB and EF; then we are to prove that they are equal in surface.

From B draw the line BD, parallel to AF; and from A draw the line AC, parallel to BE; and produce EF, if necessary, to C and D; now the parallelogram $ABDF =$ the parallelogram $ABEC$, (Th. 27). But the $\triangle$ ABE is one half the parallelogram $ABEC$, and the $\triangle$ ABF is one half the parallelogram $ABDF$; and halves of equals are equal, (Ax. 7); therefore the $\triangle$ $ABE =$ the $\triangle$ ABF.

Hence the theorem; *triangles on the same base, etc.*

THEOREM XXIX.

Parallelograms on equal bases, and between the same parallels, are equal in area.



Let $ABCD$ and $EFGH$, be two parallelograms on equal bases, AB and EF, and between the same parallels, AF and DG; then we are to prove that they are equal in area.

$AB = EF = HG$; but lines which join equal and parallel lines, are themselves equal and parallel, (Th. 26); therefore, if AH and BG be drawn, the figure $ABGH$ is a parallelogram $=$ to the parallelogram $ABCD$, (Th. 27); and if we turn the whole figure over, the two parallelograms, $GHEF$ and $GHAB$, will stand on the same base, GH, and between the same parallels; therefore, $GHEF = GHAB$, and consequently $ABCD = EFGH$, (Ax. 1).

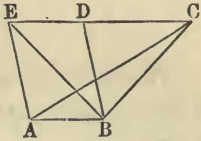
Hence the theorem; *Parallelograms on equal bases, etc.*

Cor. Triangles on equal bases, and between the same parallels, are equal in area. For, draw BD and EG; the $\triangle\ ABD$ is one half of the parallelogram AC, and the $\triangle\ EFG$ is one half of the equivalent parallelogram FH; therefore, the $\triangle\ ABD =$ the $\triangle\ EFG$, (Ax. 7).

THEOREM XXX.

If a triangle and a parallelogram are upon the same or equal bases, and between the same parallels, the triangle is equivalent to one half the parallelogram.

Let ABC be a $\triangle$, and $ABDE$ a parallelogram, on the same base AB, and between the same parallels; then we are to prove that the $\triangle\ ABC$ is equivalent to one half of the parallelogram $ABDE$.



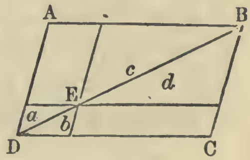
Draw EB the diagonal of the parallelogram; now, because the two $\triangle$'s ABC and ABE are on the same base, and between the same parallels, they are equivalent, (Th. 28); but the $\triangle\ ABE$ is one half the parallelogram $ABDE$, (Th. 25, Cor.); therefore the $\triangle\ ABC$ is equivalent to one half of the same parallelogram, (Ax. 7).

Hence the theorem; *if a triangle and a parallelogram, etc*

THEOREM XXXI.

The complementary parallelograms described about any point in the diagonal of any parallelogram, are equivalent to each other.

Let AC be a parallelogram, and BD its diagonal; take any point, as E, in the diagonal, and through this point draw lines parallel to the sides of the parallelogram, thus forming four parallelograms.



We are now to prove that the complementary parallelograms, AE and EC, are equivalent.

By (Th. 25, Cor.) we learn that the $\triangle\ ABD = \triangle\ DBC$. Also by the same Cor., $\triangle\ a = \triangle\ b$, and $\triangle\ c = \triangle\ d$; therefore by addition

$$\triangle\ a + \triangle\ c = \triangle\ b + \triangle\ d.$$

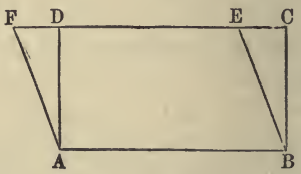
Now, from the whole $\triangle\ ABD$ take $\triangle\ a + \triangle\ c$, and from the whole $\triangle\ DBC$ take the equal sum, $\triangle\ b + \triangle\ d$, and the remaining parallelograms AE and EC are equivalent, (Ax. 3).

Hence the theorem; *the complementary parallelograms, etc.*

THEOREM XXXII.

The perimeter of a rectangle is less than that of any rhomboid standing on the same base, and included between the same parallels.

Let $ABCD$ be a rectangle, and $ABEF$ a rhomboid having the same base, and their opposite sides in the same line parallel to the base.



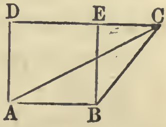
We are now to prove that the perimeter $ABCDA$ is less than $ABEFA$.

Because AD is a perpendicular from A to the line DE, and AF an oblique line, AD is less than AF, (Th. 23). For the same reason BC is less than BE; hence $AD + BC < AF + BE$. Adding the sum, $AB + DC$, to the first member of this inequality, and its equal $AB + FE$ to the second member, we have $AB + BC + CD + DA$, or the perimeter of the rectangle, less than $AB + BE + EF + FA$, or the perimeter of the rhomboid. Hence the theorem; *the perimeter of a rectangle*, etc.

Thus far, areas have been considered only relatively and in the abstract. We will now explain how we may pass to the absolute measures, or, more properly, to the numerical expressions for areas.

THEOREM XXXIII.

The area of any plane triangle is measured by the product of its base by one half its altitude; or by one half of the product of its base by its altitude.

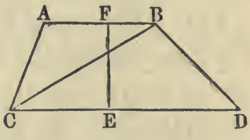


Let ABC represent any triangle, AB its base, and AD, at right angles to AB, its altitude; now we are to show that the area of ABC is equal to the product of AB by one half of AD; or one half of AB by AD; or one half of the product of AB by AD.

On AB construct the rectangle $ABED$; and the area of this rectangle is measured by AB into AD (Def. 54); but the area of the $\triangle$ ABC is equivalent to one half this rectangle, (Th. 30). Therefore, the area of the $\triangle$ is measured by $\frac{1}{2}$ $AB \times AD$, or one half the product of its base by its altitude. Hence the theorem; *the area of any plane triangle, etc.*

THEOREM XXXIV.

The area of a trapezoid is measured by one half the sum of its parallel sides multiplied by the perpendicular distance between them.



Let $ABDC$ represent any trapezoid; draw the diagonal BC, dividing it into two triangles, ABC and BCD: CD is the base of one triangle, and AB may be considered as the base of the other; and EF is the common altitude of the two triangles.

Now, by Th. 33, the area of the triangle $BCD = \frac{1}{2}$ $CD \times EF$; and the area of the $\triangle$ $ABC = \frac{1}{2}AB \times EF$; but by addition, the area of the two $\triangle$'s, or of the trapezoid, is equal to $\frac{1}{2}(AB+CD)\times EF$. Hence the theorem; *the area of a trapezoid, etc.*

THEOREM XXXV.

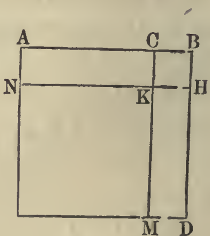
If one of two lines is divided into any number of parts, the rectangle contained by the two lines is equal to the sum of the several rectangles contained by the undivided line and the several parts of the divided line.

Let AB and AD be two lines, and suppose AB divided into any number of parts at the points E, F, G, etc.; then the whole rectangle contained by the two lines is AH, which is measured by AB into AD. But the rectangle AL is measured by AE into AD; the rectangle EK is measured by EF into EL, which is equal to EF into AD; and so of all the other partial rectangles; and the truth of the proposition is as obvious as that a whole is equal to the sum of all its parts. Hence the theorem; *if one of two lines is divided, etc.*

Aug. 8th.

THEOREM XXXVI.

If a straight line is divided into any two parts, the square described on the whole line is equivalent to the sum of the squares described on the two parts plus twice the rectangle contained by the parts.



Let AB be any line divided into any two parts at the point C; now we are to prove that the square on AB is equivalent to the sum of the squares on AC and CB plus twice the rectangle contained by AC and CB.

On AB describe the square AD. Through the point C draw CM, parallel to BD; take $BH = BC$, and through H draw HKN, parallel to AB. We now have CH, the square on CB, by direct construction.

As $AB = BD$, and $CB = BH$, by subtraction, $AB - CB = BD - BH$; or $AC = HD$. But $NK = AC$, being opposite sides of a parallelogram; and for the same reason, $KM = HD$. Therefore, (Ax. 1), $NK = KM$, and the figure NM is a square on NK, equal to a square on AC. But the whole square on AB is composed of the two squares CH, NM, and the two complements or rectangles AK and KD; and since each of these latter is AC in length, and BC in width, each has for its measure AC into CB; therefore the whole square on AB is equivalent to $\overline{AC}^2 + \overline{BC}^2 + 2AC \times CB$.

Hence the theorem; *if a straight line is divided into any two parts, etc.*

This theorem may be proved algebraically, thus:

Let w represent any whole right line divided into any two parts a and b; then we shall have the equation

$$w = a + b$$

By squaring, $w^2 = a^2 + b^2 + 2ab$.

Cor. If $a = b$, then $w^2 = 4a^2$; that is, the square described on any line is four times the square described on one half of it.

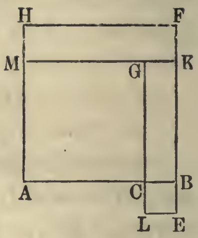
THEOREM XXXVII.

The square described on the difference of two lines is equivalent to the sum of the squares described on the two lines diminished by twice the rectangle contained by the lines.

Let AB represent the greater of two lines, CB the less line, and AC their difference.

We are now to prove that the square described on AC is equivalent to the sum of the squares on AB and BC diminished by twice the rectangle contained by AB and BC.

Conceive the square AF to be described on AB, and



the square BL on CB; on AC describe the square $ACGM$, and produce MG to K.

As $GC = AC$, and $CL = CB$, by addition, $(GC + CL)$, or GL, is equal to $AC + CB$, or AB. Therefore, the rectangle GE is AB in length, and CB in width, and is measured by $AB \times BC$.

Also $AH = AB$, and $AM = AC$; by subtraction, $MH = CB$; and as $MK = AB$, the rectangle HK is AB in length, and CB in width, and is measured by $AB \times BC$; and the two rectangles GE and HK are together equivalent to $2AB \times BC$.

Now, the squares on AB and BC make the whole figure $AHFELC$; and from this whole figure, or these two squares, take away the two rectangles HK and GE, and the square on AC only will remain; that is,

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 - 2AB \times BC.$$

Hence the theorem; *the square described on the difference of two lines, etc.*

This theorem may be proved algebraically, thus:

Let a represent the greater of two lines, b the less, and d their difference; then we must have this equation:

$$d = a - b$$

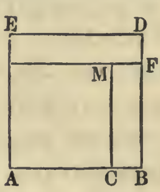
By squaring, $d^2 = a^2 + b^2 - 2ab$.

Cor. If $d = b$, then $d = \frac{a}{2}$, and $d^2 = \frac{a^2}{4}$; that is, the square described on one half of any line is equivalent to one fourth of the square described on the whole line.

THEOREM XXXVIII.

The difference of the squares described on any two lines is equivalent to the rectangle contained by the sum and difference of the lines.

Let AB be the greater of two lines, and AC the less, and on these lines describe the squares AD, AM; then, the difference of the squares on AB and AC is the two rectangles EF and FC. We are now to show that the measure of these rectangles may be expressed by $(AB + AC) \times (AB - AC)$.



The length of the rectangle EF is ED, or its equal AB; and the length of the rectangle FC is MC, or its equal AC; therefore, the length of the two together (if we conceive them put between the same parallel lines) will be $AB + AC$; and the common width is CB, which is equal to $AB - AC$; therefore, $\overline{AB}^2 - \overline{AC}^2 = (AB + AC) \times (AB - AC)$.

Hence the theorem; *the difference of the squares described on any two lines, etc.*

This theorem may be proved algebraically: thus,

Let a represent one line, and b another;

Then $a + b$ is their sum, and $a - b$ their difference;

and $$(a + b) \times (a - b) = a^2 - b^2.$$

THEOREM XXXIX.

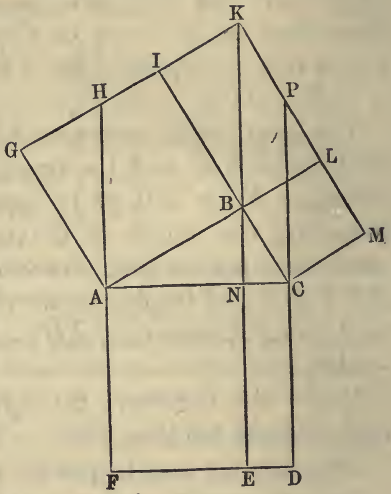
The square described on the hypotenuse of any right-angled triangle is equivalent to the sum of the squares described on the other two sides.

Let ABC represent any right-angled triangle, the right angle at B; we are to prove that the square on AC is equivalent to the sum of two squares; one on AB, the other on BC.

On the three sides of the triangle describe the three squares, AD, AI, and BM. Through the point B, draw BNE perpendicular to AC, and produce it to meet the line GI in K; also produce AF to meet GI in H, and ML to meet GI produced in K.

REMARK.—That the lines, GI and ML, produced, meet at the point K, may be readily shown. As the proof of this fact is not necessary for the demonstration, it is left as an exercise for the learner.

The angle BAG is a right angle, and the angle NAH is also a right angle; if from these equals we subtract the common angle BAH, the remaining angle, BAC, must be equal to the remaining angle GAH. The angle G is a right angle, equal to the angle ABC; and $AB = AG$; therefore, the two △'s ABC and AGH are equal, and $AH = AC$. But $AC = AF$; therefore, $AH = AF$. Now, the two



parallelograms, AE and $AHKB$ are equivalent, because they are upon equal bases, and between the same parallels, FH and EK, (Th. 29).

But the square AI, and the parallelogram $AHKB$, are equivalent, because they are on the same base, AB, and between the same parallels, AB and GK; therefore, the square AI, and the parallelogram AE, being each equivalent to the same parallelogram $AHKB$, are equivalent to each other, (Ax. 1). In the same manner we may prove that the square BM is equivalent to the rectangle ND; therefore, by addition, the two squares, AI and BM, are equivalent to the two parallelograms, AE and ND, or to the square AD.

Hence the theorem; *the square described on the hypotenuse of a right-angled triangle, etc.*

Cor. If two right-angled triangles have the hypotenuse, and a side of the one equal to the hypotenuse and a side of the other, each to each, the two triangles are equal.

Let ABC and AGH be the two $\triangle$'s, in which we suppose $AC = AH$, and $BC = GH$; then will $AG = AB$

For, we have $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$,

or, by transposing, $\overline{AC}^2 - \overline{BC}^2 = \overline{AB}^2$,

and $\overline{AH}^2 = \overline{AG}^2 + \overline{GH}^2$,

or, by transposing, $\overline{AH}^2 - \overline{GH}^2 = \overline{AG}^2$.

But by the hypothesis $\overline{AC}^2 - \overline{BC}^2 = \overline{AH}^2 - \overline{GH}^2$;

hence, $\overline{AB}^2 = \overline{AG}^2$, or, $AB = AG$.

SCHOLIUM.—The two sides, AB and BC, may vary, while AC remains constant. AB may be equal to BC; then the point N will be in the middle of AC. When AB is very near the length of AC, and BC very small, then the point N falls very near to C. Now as AE and ND are right-angled parallelograms, their areas are measured by the product of their bases by their altitudes; and it is evident that, as they have the same altitude, these areas will vary directly as their bases AN and NC; hence the squares on AB and BC, which are equivalent to those rectangles, vary as the lines AN and NC.

The following outline of the demonstration of this proposition is presented as a useful disciplinary exercise for the student.

We employ the same figure, in which no change is made except to draw through C the line CP, parallel to BK.

The first step is to prove the equality of the triangles AGH and ABC, whence $AH = AC$. But $AC = AF$; therefore $AH = AF$.

The parallelograms $AFEN$ and $AHKB$ are equivalent. Also, the parallelogram $AHKB =$ the square $ABIG$, (Th. 27), and the parallelogram $KBCP = NEDC =$ square $BCML$. Now, by adding the equals

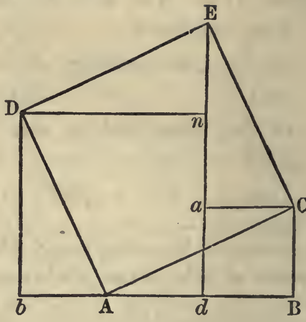
$$AFEN = ABIG$$
$$NEDC = BCML$$

we obtain $AFDC = ABIG + BCML$.

That is, the square on AC is equivalent to the sum of the squares on AB and BC.

The great practical importance of this theorem, in the extent and variety of its applications, and the frequency of its use in establishing subsequent propositions, renders it necessary that the student should master it completely. To secure this end, we present a

Second Demonstration.



Let ABC be a triangle right-angled at B. On the hypotenuse AC, describe the square $ACED$. From D and E let fall the perpendiculars Db and Ed, on AB and AB produced. Draw Dn and Ca, making right angles with Ed.

We give an outline only of the demonstration, requiring the pupil to make it complete.

First Part.—Prove the four triangles ABC, AbD, DnE, and EaC, equal to each other.

The proof is as follows: The $\triangle$'s ABC and DnE are equal, because the angles of the one are equal to the angles of the other, each to each, and the hypotenuse AC of the one, is equal to the hypotenuse DE of the other. In like manner, it may be shown that the $\triangle$'s AbD and EaC are equal.

Now, the sum of the three angles about A, is equal to the sum of the three angles of the $\triangle\ ABC$; and if, from the first sum, we take $\llcorner DAC + \llcorner CAB$, and from the second we take $\llcorner B + \llcorner CAB = \llcorner DAC + \llcorner CAB$, the remaining angles are equal; that is, $\llcorner DAb$ is equal to $\llcorner ACB$; hence the $\triangle$'s ABC and DbA have their angles equal, each to each; and since $AC = DA$, the $\triangle$'s are themselves equal, and the four triangles ABC, AbD, DnE, and EaC, are equal to each other.

Second.—Prove that the square $bDnd$ is equal to a square on AB. The square $BdaC$ is obviously on BC.

Third.—The area of the whole figure is equal to the square on AC, and the area of two of the four equal right-angled triangles.

Also, the area of the whole figure is equal to two other squares, $bDnd$ and $daCB$, and two of the four equal triangles, DnE and EaC.

Omitting or subtracting the areas of two of the four right-angled △'s from each of the two expressions for the area of the whole figure, there will remain the square on AC, equal to the sum of the two squares, $Dndb$ and $daCB$.

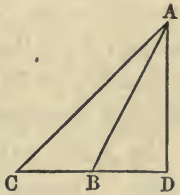
That is, $\overline{AB}^2 + \overline{BC}^2 = \overline{AC}^2$.

Hence the theorem; *the square described on the hypotenuse of a right-angled triangle, etc.*

SCHOLIUM.—Hence, to find the hypotenuse of a right-angled triangle, *extract the square root of the sum of the squares of the two sides about the right angle.*

THEOREM XL.

In any obtuse-angled triangle, the square on the side opposite the obtuse angle is greater than the sum of the squares on the other two sides, by twice the rectangle contained by either side about the obtuse angle, and the part of this side produced to meet the perpendicular drawn to it from the vertex of the opposite angle.



Let ABC be any triangle in which the angle at B is obtuse. Produce either side about the obtuse angle, as CB, and from A draw AD perpendicular to CB, meeting it produced at D.

It is obvious that $CD = CB + BD$.

By Th. 36 we have, $\overline{CD}^2 = \overline{CB}^2 + 2CB \times BD + \overline{BD}^2$,

Adding $\overline{AD}^2$ to each member of this equation, we have

$$\overline{AD}^2 + \overline{CD}^2 = \overline{CB}^2 + \overline{BD}^2 + \overline{AD}^2 + 2CB \times BD.$$

But, (Th. 39), the first member of the last equation is equal to $\overline{AC}^2$, and

$$\overline{BD}^2 + \overline{AD}^2 = \overline{AB}^2.$$

Therefore, this equation becomes

$$\overline{AC}^2 = \overline{CB}^2 + \overline{AB}^2 + 2CB \times BD.$$

That is, the square on AC is equivalent to the sum of the squares on CB and AB, increased by twice the rectangle contained by CB and BD.

Hence the theorem; *in any obtuse-angled triangle, the square on the side opposite the obtuse angle, etc.*

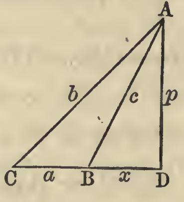
Scholium.—Conceive AB to turn about the point A, its intersection with CD gradually approaching D. The last equation above will be true, however near this intersection is to D, and when it falls upon D the triangle becomes right-angled.

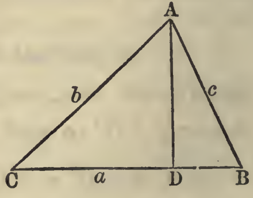
In this case the line BD reduces to zero, and the equation becomes $\overline{AC}^2 = \overline{CB}^2 + \overline{AB}^2$, in which CB and AB are now the base and perpendicular of a right-angled triangle. This agrees with Theorem 39, as it should, since we used the property of the right-angled triangle established in Theorem 39 to demonstrate this proposition; and in the equation which expresses a property of the obtuse-angled triangle, we have introduced a supposition which changes it into one which is right-angled.

THEOREM XLI.

In any triangle, the square on a side opposite an acute angle is less than the sum of the squares on the other two sides, by twice the rectangle contained by either of these sides, and the distance from the vertex of the acute angle to the foot of the perpendicular let fall on this side, or side produced, from the vertex of its opposite angle.

Let ABC, either figure, represent any triangle; C an acute angle, CB the base, and AD the perpendicular, which falls either without or on the base. Now we are to prove that





$$\overline{AB}^2 = \overline{CB}^2 + \overline{AC}^2 - 2CB \times CD.$$

From the first figure we get $BD = CD - CB$ (1)
and from the second $BD = CB - CD$ (2)

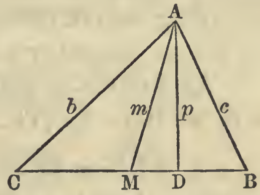
Either one of these equations will give, (Th. 37),

$$\overline{BD}^2 = \overline{CD}^2 + \overline{CB}^2 - 2CD \times CB.$$

Adding $\overline{AD}^2$ to each member and reducing, we obtain, (Th. 39), $\overline{AB}^2 = \overline{AC}^2 + \overline{CB}^2 - 2CB \times CD$, which proves the proposition. Hence the theorem.

THEOREM XLII.

If in any triangle a line be drawn from any angle to the middle of the opposite side, twice the square of this line, together with twice the square of one half the side bisected, will be equivalent to the sum of the squares of the other two sides



Let ABC be a triangle, and M the middle point of its base.

Then we are to prove that $2\overline{AM}^2 + 2\overline{CM}^2 = \overline{AC}^2 + \overline{AB}^2$.

Draw AD perpendicular to the base, and make $AD = p$, $AC = b$, $AB = c$, $CB = 2a$, $AM = m$, and $MD = x$; then $CM = a$, $CD = a + x$, $DB = a - x$.

Now by, (Th. 39), we have the two following equations:

$$p^2 + (a - x)^2 = c^2 \qquad (1)$$
$$p^2 + (a + x)^2 = b^2 \qquad (2)$$

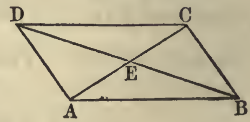
By addition, $2p^2 + 2x^2 + 2a^2 = b^2 + c^2$. But $p^2 + x^2 = m^2$.

Therefore, $2m^2 + 2a^2 = b^2 + c^2$.

This equation is the algebraic enunciation of the theorem.

THEOREM XLIII.

The two diagonals of any parallelogram bisect each other; and the sum of their squares is equivalent to the sum of the squares of the four sides of the parallelogram.



Let $ABCD$ be any parallelogram, and AC and BD its diagonals.

We are now to prove,

1st. That $AE = EC$, and $DE = EB$.

2d. That $\overline{AC}^2 + \overline{BD}^2 = \overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{AD}^2$.

1. The two triangles ABE and CDE are equal, because $AB = CD$, the angle $ABE =$ the alternate angle CDE, and the vertical angles at E are equal; therefore, AE, the side opposite the angle ABE, is equal to CE, the side opposite the equal angle CDE; also EB, the remaining side of the one $\triangle$, is equal to ED, the remaining side of the other triangle.

2. As ACD is a triangle whose base, AC, is bisected in E, we have, by (Th. 42),

$$2\overline{AE}^2 + 2\overline{ED}^2 = \overline{AD}^2 + \overline{DC}^2 \quad (1)$$

And as ACB is a triangle whose base, AC, is bisected in E, we have

$$2\overline{AE}^2 + 2\overline{EB}^2 = \overline{AB}^2 + \overline{BC}^2 \quad (2)$$

By adding equations (1) and (2), and observing that $\overline{EB}^2 = \overline{ED}^2$, we have

$$4\overline{AE}^2 + 4\overline{ED}^2 = \overline{AD}^2 + \overline{DC}^2 + \overline{AB}^2 + \overline{BC}^2$$

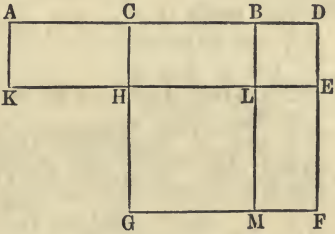
But, four times the square of the half of a line is equivalent to the square of the whole line, (Th. 36, Corollary); therefore $4\overline{AE}^2 = \overline{AC}^2$, and $4\overline{ED}^2 = \overline{DB}^2$; and by substituting these values, we have

$$\overline{AC}^2 + \overline{BD}^2 = \overline{AB}^2 + \overline{BC}^2 + \overline{DC}^2 + \overline{AD}^2,$$

which equation conforms to the enunciation of the theorem.

THEOREM XLIV.

If a line be bisected and produced, the rectangle contained by the whole line and the part produced, together with the square of one half the bisected line, will be equivalent to the square on a line made up of the part produced and one half the bisected line.



Let AB be any line, bisected in C and produced to D. On CD describe the square CF, and on BD describe the square BE.

The sides of the square BE being produced, the square GL will be formed. Also, complete the construction of the rectangle $ADEK$.

Then we are to prove that the rectangle, AE, and the square, GL, are together equivalent to the square, $CDFG$.

The two complementary rectangles, CL and LF, are equal, (Th. 31). But $CL=AH$, the line AB being bisected at C; therefore AL is equal to the sum of the two complementary rectangles of the square CF. To AL add the square BE, and the whole rectangle, AE, will be equal to the two rectangles CE and EM. To each of these equals add HM, or the square on HL or its equal CB, and we have rectangle $AE +$ square $HM = \overline{CD}^2$; but rectangle $AE = AD \times BD$, and square $HM = \overline{CB}^2$. Hence the theorem, etc.

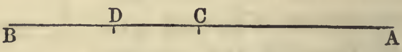
Scholium.—If we represent AB by $2a$, and BD by x, then $AD = 2a + x$, and $AD \times BD = 2ax + x^2$. But $\overline{CB}^2 = a^2$; adding this equation to the preceding, member to member, we get $AD \times BD + \overline{CB}^2 = a^2 + 2ax + x^2 = \overline{a+x}^2$. But $CD = a + x$; hence this equation is equivalent to the equation $AD \times DB + \overline{CB}^2 = \overline{CD}^2$, which is the algebraic proof of the theorem.

THEOREM XLV.

If a straight line be divided into two equal parts, and also into two unequal parts, the rectangle contained by the two unequal parts together with the square of the line between the points of division, will be equivalent to the square on one half the line.

Let AB be a line bisected in C, and divided into two unequal parts in D.

We are to prove that $AD \times DB + \overline{CD}^2 = \overline{AC}^2$, or $\overline{CB}^2$.



We see by inspection that $AD = AC + CD$, and $BD = AC - CD$; therefore by (Th. 38), we have

$$AD \times BD = \overline{AC}^2 - \overline{CD}^2.$$

By adding $\overline{CD}^2$ to each of these equals, we obtain

$$AD \times BD + \overline{CD}^2 = \overline{AC}^2$$

Hence the theorem.

BOOK II.

PROPORTION.

DEFINITIONS AND EXPLANATIONS.

THE word Proportion, in its common meaning, denotes that *general relation* or symmetry existing between the different parts of an object which renders it agreeable to our taste, and conformable to our ideas of beauty or utility; but in a mathematical sense,

1. Proportion is the numerical relation which one quantity bears to another of the same kind.

As the magnitudes compared must be of the same kind, proportion in geometry can be only that of *a line to a line, a surface to a surface, an angle to an angle*, or *a volume to a volume.*

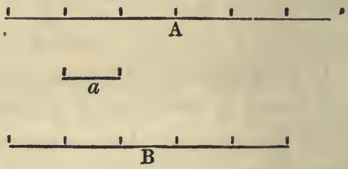
2. Ratio is a term by which the number which measures the proportion between two magnitudes is designated, and is the quotient obtained by dividing the one by the other. Thus, the ratio of A to B is $\frac{B}{A}$, or $A : B$, in which A is called the *antecedent*, and B the *consequent.* If, therefore, the magnitude A be assumed as the unit or standard, this quotient is the numerical value of B expressed in terms of this unit.

It is to be remarked that this principle lies at the foundation of the method of representing quantities by numbers. For example, when we say that a body weighs twenty-five pounds, it is implied that the weight of this body has been compared, directly or indirectly, with that of the standard, one pound. And so of geometrica magnitudes; when a line, a surface, or a volume is said to be fifteen linear, superficial, or cubical feet, it is understood that it has been referred to its particular unit, and found to contain it fifteen times; that is, fifteen is the ratio of the unit to the magnitude.

When two magnitudes are referred to the same unit, the ratio of the numbers expressing them will be the ratio of the magnitudes themselves.

Thus, if A and B have a common unit, a, which is contained in A, m times, and in B, n times, then $A = ma$ and $B = na$, and $\frac{B}{A} = \frac{na}{ma} = \frac{n}{m}$.

To illustrate, let the line A contain the line a six times, and let the line B contain the same line a five times: then $A=6a$ and $B=5a$, which give $\frac{B}{A} = \frac{5a}{6a} = \frac{5}{6}$.



3. A **Proportion** is a formal statement of the equality of two ratios.

Thus, if we have the four magnitudes A, B, C and D, such that $\frac{B}{A} = \frac{D}{C}$, this relation is expressed by the proportion $A : B :: C : D$, or $A : B = C : D$, the first of which is read, A is to B as C is to D; and the second, the ratio of A to B is equal to that of C to D.

4. The **Terms** of a proportion are the magnitudes, or more properly the representatives of the magnitudes compared.

5. The **Extremes** of a proportion are its first and fourth terms.

6. The **Means** of a proportion are its second and third terms.

7. A **Couplet** consists of the two terms of a ratio. The first and second terms of a proportion are called the *first couplet*, and the third and fourth terms are called the *second couplet.*

8. The **Antecedents** of a proportion are its first and third terms.

9. The **Consequents** of a proportion are its second and fourth terms.

In expressing the equality of ratios in the form of a proportion, we may make the denominators the antecedents, and the numerators the consequents, or the reverse, without affecting the relation between the magnitudes. It is, however, a matter of some little importance to the beginner to adopt a uniform rule for writing the terms of the ratios in the proportion; and we shall always, unless otherwise stated, make the denominators of the ratios the antecedents, and the numerators the consequents.*

10. **Equimultiples** of magnitudes are the products arising from multiplying the magnitudes by the same number. Thus, the products, Am and Bm, are equimultiples of A and B.

11. **A Mean Proportional** between two magnitudes is a magnitude which will form with the two a proportion, when it is made a consequent in the first ratio, and an antecedent in the second. Thus, if we have three magnitudes A, B, and C, such that $A : B :: B : C$, B is a mean proportional between A and C.

12. Two magnitudes are *reciprocally*, or *inversely* proportional when, in undergoing changes in value, one is multiplied and the other is divided by the same number. Thus, if A and B be two magnitudes, so related that when A becomes mA, B becomes $\frac{B}{m}$, A and B are said to be inversely proportional.

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* For discussion of the two methods of expressing Ratio, see University Algebra.

13. A *Proportion is taken inversely* when the antecedents are made the consequents and the consequents the antecedents.

14. A *Proportion is taken alternately, or by alternation,* when the antecedents are made one couplet and the consequents the other.

15. **Mutually Equiangular Polygons** have the same number of angles, those of the one equal to those of the others, each to each, and the angles like placed.

16. **Similar Polygons** are such as are mutually equiangular, and have the sides about the equal angles, taken in the same order, proportional.

17. **Homologous Angles** in similar polygons are those which are equal and like placed; and

18. The **Homologous Sides** are those which are like disposed about the homologous angles.

THEOREM I.

If the first and second of four magnitudes are equal, and also the third and fourth, the four magnitudes may form a proportion.

Let A, B, C, and D represent four magnitudes, such that $A = B$ and $C = D$; we are to prove that $A : B :: C : D$.

Now, by hypothesis, A is equal to B, and their ratio is therefore 1; and since, by hypothesis, C is equal to D, their ratio is also 1.

Hence, the ratio of A to B is equal to that of C to D; and, (by Def. 3),

$$A : B :: C : D.$$

Therefore, four magnitudes which are equal, two and two, constitute a proportion.

THEOREM II.

If four magnitudes constitute a proportion, the product of the extremes is equal to the product of the means.

Let the four magnitudes A, B, C, and D form the proportion $A : B :: C : D$; we are to prove that $A \times D = B \times C$.

The ratio of A to B is expressed by $\frac{B}{A} = r$.

The ratio of C to D is expressed by $\frac{D}{C} = r$.

Hence, (Ax. 1), $\frac{B}{A} = \frac{D}{C}$.

Multiplying each of these equals by $A \times C$, we have

$$B \times C = A \times D.$$

Hence the theorem; *if four magnitudes are in proportion, etc.*

Cor. 1. Conversely; *If we have the product of two magnitudes equal to the product of two other magnitudes, they will constitute a proportion of which either two may be made the extremes and the other two the means.*

Let the magnitudes $B \times C = A \times D$. Dividing both members of the equation by $A \times C$, we obtain $\frac{B}{A} = \frac{D}{C}$.

Hence the proportion $A : B :: C : D$.

Cor. 2. If we divide both members of the equation

$$A \times D = B \times C \quad \text{by } A,$$

we have $$D = \frac{B \times C}{A}.$$

That is, to find the fourth term of a proportion, *multiply the second and third terms together and divide the product by the first term.* This is the Rule of Three of Arithmetic.

This equation shows that any one of the four terms can be found by a like process, *provided* the other three are given.

THEOREM III.

If three magnitudes are continued proportionals, the product of the extremes is equal to the square of the mean.

Let A, B, and C represent the three magnitudes:

Then $A : B :: B : C$, (by Def. 11).

But, (by Th. 2), the product of the extremes is equal to the product of the means; that is, $A \times C = B^2$.

Hence the theorem; *if three magnitudes, etc.*

THEOREM IV.

Equimultiples of any two magnitudes have the same ratio as the magnitudes themselves; and the magnitudes and their equimultiples may therefore form a proportion.

Let A and B represent two magnitudes, and mA and mB their equimultiples.

Then we are to prove that $A : B :: mA : mB$.

The ratio of A to B is $\frac{B}{A}$, and of mA to mB is $\frac{mB}{mA} = \frac{B}{A}$, the same ratio.

Hence the theorem; *equimultiples of any two magnitudes, etc.*

THEOREM V.

If four magnitudes are proportional, they will be proportional when taken inversely.

If $A : B :: mA : mB$, then $B : A :: mB : mA$;

For in either case, the product of the extremes equals that of the means; or the ratio of the couplets is the same.

Hence the theorem; *if four quantities are proportional, etc.*

THEOREM VI.

Magnitudes which are proportional to the same proportionals, are proportional to each other.

If $A : B = P : Q$ and $a : b = P : Q$ $\Big\}$ Then we are to prove that $A : B = a : b$.

From the 1st proportion, $\dfrac{B}{A} = \dfrac{Q}{P}$;

From the 2d " $\dfrac{b}{a} = \dfrac{Q}{P}$;

Therefore, by (Ax. 1), $\dfrac{B}{A} = \dfrac{b}{a}$, or $A : B = a : b$.

Hence the theorem; *magnitudes which are proportional to the same proportionals, etc.*

Cor. 1. This principle may be extended through any number of proportionals.

Cor. 2. *If the ratio of an antecedent and consequent of one proportion is equal to the ratio of an antecedent and consequent of another proportion, the remaining terms of the two proportions are proportional.*

For, if $A : B :: C : D$

and $M : N :: P : Q$

in which $\dfrac{B}{A} = \dfrac{N}{M}$, then $\dfrac{D}{C} = \dfrac{Q}{P}$;

hence $C : D :: P : Q$.

THEOREM VII.

If any number of magnitudes are proportional, any one of the antecedents will be to its consequent as the sum of all the antecedents is to the sum of all the consequents.

Let A, B, C, D, E, etc., represent the several magnitudes which give the proportions

$$A : B :: C : D$$
$$A : B :: E : F$$
$$A : B :: G : H, \text{ etc., etc}$$

To which we may annex the identical proportion,

$$A : B :: A : B.$$

Now, (by Th. 2), these proportions give the following equations,

$$A \times D = B \times C$$
$$A \times F = B \times E$$
$$A \times H = B \times G$$
$$A \times B = B \times A, \text{ etc. etc.}$$

From which, by addition, there results the equation,

$$A(B + D + F + H, \text{ etc.}) = B(A + C + E + G, \text{ etc.})$$

But the sums $B + D + F$, etc., and $A + C + E$, etc., may be separately regarded as single magnitudes; therefore, (Th. 2, Cor. 1),

$$A : B :: A + C + E + G, \text{ etc. } : B + D + F + H, \text{ etc.}$$

Hence the theorem; *if any number of magnitudes are proportional, etc.*

THEOREM VIII.

If four magnitudes constitute a proportion, the first will be to the sum of the first and second as the third is to the sum of the third and fourth.

By hypothesis, $A : B :: C : D$; then we are to prove that $A : A + B :: C : C + D$.

By the given proportion, $\frac{B}{A} = \frac{D}{C}$.

Adding unity to both members, and reducing them to the form of a fraction, we have $\frac{B + A}{A} = \frac{D + C}{C}$. Changing this equation into its equivalent proportional form, we have

$$A : A + B :: C : C + D.$$

Hence the theorem; *if four magnitudes constitute a proportion, etc.*

Cor. If we subtract each member of the equation $\frac{B}{A} =$

$\frac{D}{C}$ from unity, and reduce as before, we shall have

$$A : A - B :: C : C - D.$$

Hence also; *if four magnitudes constitute a proportion, the first is to the difference between the first and second, as the third is to the difference between the third and fourth.*

THEOREM IX.

If four magnitudes are proportional, the sum of the first and second is to their difference as the sum of the third and fourth is to their difference.

Let A, B, C, and D be the four magnitudes which give the proportion

$$A : B :: C : D;$$

we are then to prove that they will also give the proportion

$$A + B : A - B :: C + D : C - D.$$

By Th. 8 we have $A : A + B = C : C + D$.

Also by Corollary, same Th., $A : A - B = C : C - D$.

Now, if we change the order of the means in these proportions, which may be done, since the products of extremes and means remain the same, we shall have

$$A : C = A + B : C + D.$$
$$A : C = A - B : C - D.$$

Hence, (Th. 6), we have

$$A + B : C + D = A - B : C - D.$$

Or, $\quad A + B : A - B = C + D : C - D.$

Hence the theorem; *if four magnitudes are proportional, etc.*

THEOREM X.

If four magnitudes are proportional, like powers or like roots of the same magnitudes are also proportional.

If the four magnitudes, A, B, C, and D, give the proportion

$$A : B :: C : D,$$

we are to prove that

$$A^n : B^n :: C^n : D^n.$$

The hypothesis gives the equation $\frac{B}{A} = \frac{D}{C}$. Raising both members of this equation to the nth power, we have $\frac{B^n}{A^n} = \frac{D^n}{C^n}$, which, expressed in its equivalent proportional form, gives

$$A^n : B^n :: C^n : D^n.$$

If n is a *whole number*, the terms of the given proportion are each raised to a power; but if n is a fraction having unity for its numerator, and a *whole number* for its denominator, like roots of each are taken.

As the terms of the proportion may be first raised to like powers, and then like roots of the resulting proportion be taken, n may be any number whatever.

Hence the theorem; *if four magnitudes, etc.*

THEOREM XI.

If four magnitudes are proportional, and also four others, the products which arise from multiplying the first four by the second four, term by term, are also proportional.

Admitting that $A : B :: C : D,$
and $X : Y :: M : N,$
We are to show that $AX : BY :: CM : DN.$

From the first proportion, $\frac{B}{A} = \frac{D}{C}$;

From the second, $\frac{Y}{X} = \frac{N}{M}$.

Multiply these equations, member by member, and

$$\frac{BY}{AX} = \frac{DN}{CM};$$

Or, $AX : BY :: CM : DN.$

The same would be true in any number of proportions.

Hence the theorem; *if four magnitudes are, etc.*

THEOREM XII.

If four magnitudes are proportional, and also four others, the quotients which arise from dividing the first four by the second four, term by term, are proportional.

By hypothesis, $A : B :: C : D$,
and $X : Y :: M : N$.
Multiply extremes and means, $AD = CB$, (1)
and $XN = MY$. (2)
Divide (1) by (2), and $\frac{A}{X} \times \frac{D}{N} = \frac{C}{M} \times \frac{B}{Y}$.

Convert these four factors, which make two equal products, into a proportion, and we have

$$\frac{A}{X} : \frac{B}{Y} :: \frac{C}{M} : \frac{D}{N}.$$

By comparing this with the given proportions, we find it is composed of the quotients of the several terms of the first proportion, divided by the corresponding terms of the second.

Hence the theorem; *if four magnitudes are proportional, etc.*

THEOREM XIII.

If four magnitudes are proportional, we may multiply the first couplet, the second couplet, the antecedents or the consequents, or divide them by the same quantity, and the results will be proportional in every case.

Let the four magnitudes A, B, C, and D give the proportion $A : B :: C : D$. By multiplying the extremes and means we have

$$A.D = B.C \qquad (1)$$

Multiply both members of this equation by any number, as a, and we have

$$aA.D = aB.C$$

By converting this equation into a proportion in four different ways, we have as follows :

$$aA : aB :: C : D$$
$$A : B :: aC : aD$$
$$aA : B :: aC : D$$
$$A : aB :: C : aD.$$

Resuming the original equation, (1), and dividing both members by a, we have

$$\frac{A.D}{a} = \frac{B.C}{a}.$$

This equation may also be converted into a proportion in four different ways, with the following results:

$$\frac{A}{a} : \frac{B}{a} :: C : D$$
$$A : B :: \frac{C}{a} : \frac{D}{a}$$
$$\frac{A}{a} : B :: \frac{C}{a} : D$$
$$A : \frac{B}{a} :: C : \frac{D}{a}.$$

Hence the theorem; *if four magnitudes are in proportion, etc.*

THEOREM XIV.

If three magnitudes are in proportion, the first is to the third as the square of the first is to the square of the second.

Let A, B, and C, be three proportionals.

Then we are to prove that $A : C = A^2 : B^2$

By (Th. 3) $$AC = B^2$$

Multiply this equation by the numeral value of A, and we have $$A^2C = AB^2$$

This equation gives the following proportion:

$$A : C = A^2 : B^2.$$

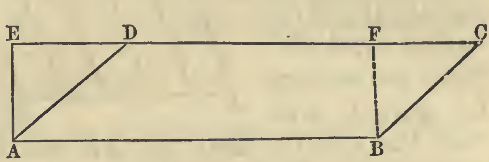
Hence the theorem.

REMARK.—It is now proposed to make an application of the preceding abstract principles of proportion, in geometrical investigations

THEOREM XV.

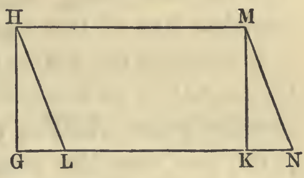
If two parallelograms are equal in area, the base and perpendicular of either may be made the extremes of a proportion, of which the base and perpendicular of the other are the means.

Let $ABCD$, and $HLNM$, be two parallelograms having equal areas, by hypothesis; then we are to prove that



$$AB : LN :: MK : BF,$$

in which MK and BF are the altitudes or perpendiculars of the parallelograms.



This proportion is true, if the product of the extremes is equal to the product of the means; that is, if the equation

$$AB.BF = LN.MK \text{ is true.}$$

But $AB.BF$ is the measure of the rectangle $ABFE$, by (Definition 54, B. I.), and this rectangle is equal in area to the parallelogram $ABCD$, (B. I., Th. 27).

In the same manner, we may prove that $LN.MK$ is the measure of the parallelogram $NLHM$. But these two parallelograms have equal areas by hypothesis.

Therefore, $AB.BF = LN.MK$ is a true equation, and Th. 2, Cor. 1), gives the proportion

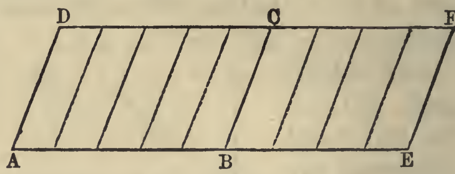
$$AB : LN :: MK : BF.$$

Hence the theorem; *if two parallelograms are equal in area, etc.*

THEOREM XVI.

Parallelograms having equal altitudes are to each other as their bases.

Since parallelograms having equal bases and equal altitudes are equal in area, however much their angles may differ, we can suppose the two parallelograms under consideration to be mutually equiangular, without in the least impairing the generality of this theorem. Therefore, let $ABCD$ and $AEFD$ be two parallelograms having equal altitudes, and let them be placed with



their bases on the same line AE, and let the side, AD, be common. First suppose their bases commensurable, and that AE being divided into nine equal parts, AB contains five of those parts.

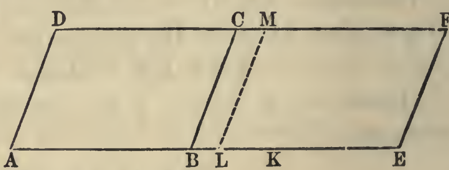
If, through the points of division, lines be drawn parallel to AD, it is obvious that the whole figure, or the parallelogram, $AEFD$, will be divided into nine equal parts, and that the parallelogram, $ABCD$, will be composed of five of those parts.

Therefore, $ABCD : AEFD :: AB : AE :: 5 : 9$.

Whatever be the whole numbers having to each other the ratio of the lines AB and AE, the reasoning would remain the same, and the proportion is established when the bases are commensurable. But if the bases are not to each other in the ratio of any two whole numbers, it remains still to be shown that

$$AEFD : ABCD :: AE : AB \quad (1)$$

If this proportion is not true, there must be a line greater or less than AB, to which AE will have the



same ratio that $AEFD$ has to $ABCD$.

Suppose the fourth proportional greater than AB, as AK, then,

$$AEFD : ABCD :: AE : AK \quad (2).$$

If we now divide the line AE into equal parts, each less than the line BK, one point of division, at least, will fall between B and K. Let L be such point, and draw LM parallel to BC.

This construction makes AE and AL commensurable; and by what has been already demonstrated, we have

$$AEFD : ALMD :: AE : AL. \quad (3)$$

Inverting the means in proportions (2) and (3), they become

$$AEFD : AE :: ABCD : AK;$$

and $$AEFD : AE :: ALMD : AL.$$

Hence, (Th. 6),

$$ABCD : AK :: ALMD : AL.$$

By inverting the means in this last proportion, we have

$$ABCD : ALMD :: AK : AL.$$

But AK is, by hypothesis, greater than AL; hence, if this proportion is true, $ABCD$ must be greater than $ALMD$; but on the contrary it is less. We therefore conclude that the supposition, that the fourth proportional, AK, is greater than AB, from which alone this absurd proportion results, is itself absurd.

In a similar manner it can be proved absurd to suppose the fourth proportional less than AB.

Therefore the fourth term of the proportion (1) can be neither less nor greater than AB; it is then AB itself, and parallelograms having equal altitudes are to each other as their bases, whether these bases are commensurable or not.

Hence the theorem; *Parallelograms having equal altitudes, etc.*

Cor. 1. Since a triangle is one half of a parallelogram having the same base as the triangle and an equal altitude, and as the halves of magnitudes have the same ratio as their wholes; therefore, *Triangles having the same or equal altitudes are to each other as their bases.*

Cor. 2. Any triangle has the same area as a right-angled triangle having the same base and an equal altitude; and as either side about the right angle of a right-angled triangle may be taken as the base, it follows that

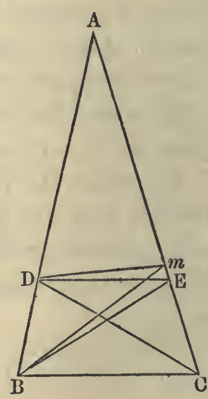
Two triangles having the same or equal bases are to each other as their altitudes.

Cor. 3. Since either side of a parallelogram may be taken as its base, it follows from this theorem that

Parallelograms having equal bases are to each other as their altitudes.

THEOREM XVII.

If lines are drawn cutting the sides, or the sides produced, of a triangle proportionally, such secant lines are parallel to the base of the triangle; and conversely, *lines drawn parallel to the base of a triangle cut the sides, or the sides produced, proportionally.*



Let ABC be any triangle, and draw the line DE dividing the sides AB and AC into parts which give the proportion

$$AD : DB :: AE : EC.$$

We are to prove that DE is parallel to BC.

If DE is not a parallel through the point D to the line BC, suppose Dm to be that parallel; and draw the lines DC and Bm.

Now, the two triangles ADm and mDC, have the same altitude, since they have a common vertex, D, and their bases in the same line, AC; hence, they are to each other as their bases, Am and mC, (Th. 16, Cor. 1).

That is, $\triangle ADm : \triangle mDC :: Am : mC$,
Also, $\triangle AmD : \triangle DmB :: AD : DB$.

But, since Dm is supposed parallel to BC, the triangles DBm and DCm have equal areas, because they are on the same base and between the same parallels, (Th. 28, B. I).

Therefore the terms of the first couplets in the two preceding proportions are equal each to each, and consequently the terms of the second couplets are proportional, (Theorem 6).

That is, $AD : DB :: Am : mC$
But $AD : DB :: AE : EC$ by hypothesis.

Hence we again have two proportions having the first couplets, the same in both, and we therefore have

$$AE : EC :: Am : mC$$

By alternation this becomes

$$AE : Am :: EC : mC$$

That is, AE is to Am, a greater magnitude is to a less, as EC is to mC, a less to a greater, which is absurd. Had we supposed the point m to fall between E and C, our conclusion would have been equally absurd; hence the suppositions which have led to these absurd results are themselves absurd, and the line drawn through the point D parallel to BC must intersect AC in the point E. Therefore the parallel and the line DE are one and the same line.

Conversely: If DE be drawn parallel to the base of the triangle, then will

$$AD : DB :: AE : EC$$

For as before,

$$\triangle ADE : \triangle EDC :: AE : EC$$

and $\triangle DEB : \triangle ADE :: DB : AD$

Multiplying the corresponding terms of these proportions, and omitting the common factor, $\triangle ADE$, in the first couplet, we have

$$\triangle DEB : \triangle EDC :: AE \times DB : EC \times AD.$$

But the $\triangle$'s DEB and EDC have equal areas, (Th. 28, B. I); hence $AE \times DB = EC \times AD$, which in the form of a proportion is

$$AE : EC :: AD : DB$$

or, $$AD : DB :: AE : EC$$

and therefore the line parallel to the base of the triangle, divides the sides proportionally.

It is evident that the reasoning would remain the same, had we conceived ADE to be the triangle and the sides to be produced to the points B and C.

Hence the theorem; *if lines are drawn cutting the sides, etc.*

Cor. 1. Because DE is parallel to BC, and intersects the sides AB and AC, the angles ADE and ABC are equal. For the same reason the angles AED and ACB are equal, and the $\triangle$'s ADE and ABC are equiangular.

Let us now take up the triangle ADE, and place it on ABC; the angle ADE falling on $\angle B$, the side AD on the side AB, and the side DE on the side BC

Now, since the angle A is common, and the angles AED and ACB are equal, the side AE of the $\triangle ADE$, in its new position, will be parallel to the side AC of the $\triangle ABC$.

The last proportion of this Th. gives (Th. 8 and Th. 5),

$$AD : AE :: AB : AC$$

From the above construction we obtain, by a similar course of reasoning, the proportion

$$AD : DE :: AB : BC$$

And in like manner it may be shown that

$$AE : ED :: AC : CB$$

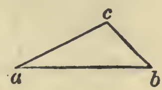
That is, *the sides about the equal angles of equiangular triangles, taken in the same order, are proportional, and the triangles are similar*, (Def. 16).

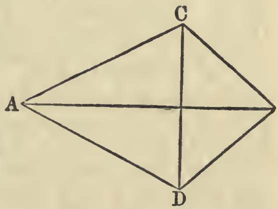
Cor. 2. *Two triangles having an angle in one equal to an angle in the other, and the sides about these equal angles proportional, are equiangular and similar.*

For, if the smaller triangle be placed on the larger, the equal angles of the triangles coinciding, then will the sides opposite these angles be parallel, and the triangles will therefore be equiangular and similar.

THEOREM XVIII.

If any triangle have its sides respectively proportional to the like or homologous sides of another triangle, each to each, then the two triangles will be equiangular and similar.





Let the triangle abc have its sides proportional to the triangle ABC; that is, ac to AC as cb to CB, and ac to AC as ab to AB; then we are to prove that the $\triangle$'s, abc and ABC, are equiangular and similar.

On the other side of the base, AB, and from A, conceive the angle BAD to be drawn = to the $\angle\ a$; and from the point B, conceive the angle ABD to be drawn = to the $\angle\ b$. Then the third $\angle\ D$ must be = to the third $\angle\ c$, (B. I, Th. 12, Cor. 2); and the $\triangle\ ABD$ will be equiangular to the $\triangle\ abc$ by construction.

Therefore, $\quad ac : ab = AD : AB$

By hypothesis, $ac : ab = AC : AB$

Hence, $\quad AD : AB = AC : AB$, (Th. 6).

In this last proportion the consequents are equal; therefore, the antecedents are equal: that is,

$$AD = AC$$

In the same manner we may prove that

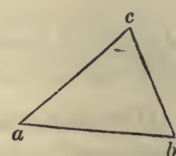
$$BD = CB$$

But AB is common to the two triangles; therefore, the three sides of the $\triangle\ ABD$ are respectively equal to the three sides of the $\triangle\ ABC$, and the two $\triangle$'s are equal, (B. I, Th. 21).

But the $\triangle$'s ABD, and abc, are equiangular by construction; therefore, the $\triangle$'s, ABC, and abc, are also equiangular and similar.

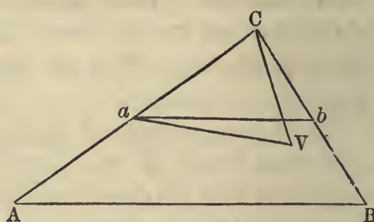
Hence the theorem; *if any triangle have its sides, etc.*

Second Demonstration.



Let abc and ABC be two triangles whose sides are respectively proportional, then will the triangles be equiangular and similar.

That is, $\angle a = \angle A$, $\angle b = \angle B$, and $\angle c = \angle C$.



If the $\angle c$ be in fact equal to the $\angle C$, the triangle abc can be placed on the triangle ABC, ca taking the direction of CA and cb of CB. The line ab will then divide the sides CA and CB proportionally, and will therefore be parallel to AB, and the triangles will be equiangular and similar, (Th. 17).

But if the $\angle c$ be not equal to the $\angle C$, then place ac on AC as before, the point c falling on C. Under the present supposition cb will not fall on CB, but will take another direction, CV, on one side or the other of CB Make CV equal to cb and draw aV.

Now, the $\triangle\ abc$ is represented in magnitude and position by the $\triangle\ aVC$; and if, through the point a, the line ab be drawn parallel to AB, we shall have

$$Ca : CA :: ab : AB;$$

but by (Hy.) $\quad Ca : CA :: aV : AB.$

Hence, (Th. 6),

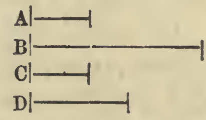
$$ab : AB :: aV : AB;$$

which requires that $ab = aV$, but (Th. 22, B. 1) ab can not be equal to aV; hence the last proportion is absurd, and the supposition that the $\angle c$ is not equal to the $\angle C$, which leads to this result, is also absurd. Therefore, the $\angle c$ is equal to the $\angle C$, and the triangles are equiangular and similar.

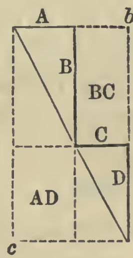
Hence the theorem; *if any triangle have its sides, etc.*

THEOREM XIX.

If four straight lines are in proportion, the rectangle contained by the lines which constitute the extremes, is equivalent to that contained by those which constitute the means of the proportion.



Let A, B, C, D, represent the four lines; then we are to show, geometrically, that $A \times D = B \times C$.



Place A and B at right angles to each other, and draw the hypotenuse. Also place C and D at right angles to each other, and draw the hypotenuse. Then bring the two triangles together, so that C shall be at right angles to B, as represented in the figure.

Now, these two $\triangle$'s have each a R. $\angle$, and the sides about the equal angles are proportional; that is, $A : B :: C : D$; hence, (Th. 17, Cor. 2), the two $\triangle$'s are equiangular, and the acute angles which meet at the extremities of B and C, are together equal to one right angle, and the lines B and C are so placed as to make another right angle; therefore, also, the extremities of A, B, C, and D, are in one right line, (Th. 3, B. I), and that line is the diagonal of the parallelogram *bc*. By Th. 31, B. I, the complementary parallelograms about this diagonal are equal; but, one of these parallelograms is B in length, and C in width, and the other is D in length and A in width; therefore,

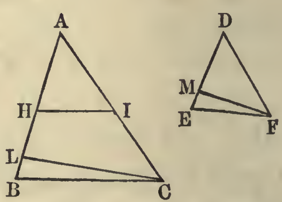
$$B \times C = A \times D.$$

Hence the theorem; *if four straight lines are in proportion, etc.*

Cor. When $B = C$, then $A \times D = B^2$, and B is the mean proportional between A and D. That is, if three straight lines are in proportion, the rectangle contained by the first and third lines is equivalent to the square described on the second line.

THEOREM XX.

Similar triangles are to one another as the squares of their homologous sides.



Let ABC and DEF be two similar triangles, and LC and MF perpendiculars to the sides AB and DE respectively. Then we are to prove that

$\triangle ABC : \triangle DEF = AB^2 : DE^2$.

By the similarity of the triangles, we have,

$$AB : DE = LC : MF$$

But, $\quad AB : DE = AB : DE$

Hence, $\quad \overline{AB}^2 : \overline{DE}^2 = AB \times LC : DE \times MF.$

But, (by Th. 30, B. I), $AB \times LC$ is double the area of the $\triangle\ ABC$, and $DE \times MF$ is double the area of the $\triangle\ DEF$.

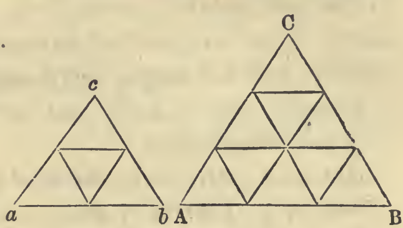
Therefore, $\quad \triangle ABC : \triangle DEF :: AB \times LC : DE \times MF$

And, (Th. 6), $\triangle ABC : \triangle DEF = \overline{AB}^2 : \overline{DE}^2$.

Hence the theorem; *similar triangles are to one another, etc.*

The following illustration will enable the learner fully to comprehend this important theorem, and it will also serve to impress it upon his memory.

Let abc and ABC represent two equiangular triangles. Suppose the length of the side ac to be two units, and the length of the corresponding side AC to be three units.



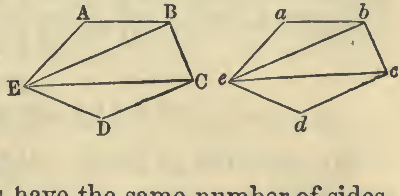
Now, drawing lines through the points of division of the sides ac and AC, parallel to the other sides of the triangles, we see that the smaller triangle is composed of four equal triangles, while the larger contains nine such triangles. That is,

the sides of the triangles are as $2 : 3$,
and their areas are as $4 : 9 = 2^2 : 3^2$.

THEOREM XXI.

Similar polygons may be divided into the same number of triangles; and to each triangle in one of the polygons there will be a corresponding triangle in the other polygon, these triangles being similar and similarly situated.

Let $ABCDE$ and $abcde$ be two similar polygons. Now it is obvious that we can divide each polygon into as many triangles as the figure has sides, *less two;* and as the polygons have the same number of sides, the diagonals drawn from the vertices of the homologous angles will divide them into the same number of triangles.



Since the polygons are similar, the angles EAB and eab, are equal, and

$$EA : AB :: ea : ab.$$

Hence the two triangles, EAB and eab, having an angle in the one equal to an angle in the other, and the sides about these angles proportional, are equiangular and similar, and the angles ABE and abe are equal.

But the angles ABC and abc are equal, because the polygons are similar.

Hence, $\angle ABC - \angle ABE = \angle abc - \angle abe$;

that is, $\angle EBC = \angle ebc$.

The triangles, EAB and eab, being similar, their homologous sides give the proportion,

$$AB : BE :: ab : be; \quad (1)$$

and since the polygons are similar, the sides about the equal angles B and b are proportional, and we have

$$AB : BC :: ab : bc;$$

or, $$BC : AB :: bc : ab. \quad (2)$$

Multiplying proportions (1) and (2), term by term, and omitting in the result the factor AB common to the terms of the first couplet, and the factor ab common to the terms of the second, we have

$$BC : BE :: bc : be.$$

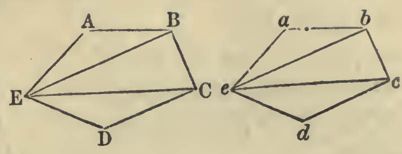
Hence the $\triangle$'s EBC and ebc are equiangular and similar; and thus we may compare all of the triangles of one polygon with those like placed in the other.

Hence the theorem; *similar polygons may be divided, etc*

THEOREM XXII.

The perimeters of similar polygons are to one another as their homologous sides; and their areas are to one another as the squares of their homologous sides.

Let $ABCDE$ and $abcde$ be two similar polygons; then we are to prove that AB is to the sum of all the sides of the polygon $ABCD$, as ab is to the sum of all the sides of the polygon $abcd$.



We have the identical proportion

$$AB : ab :: AB : ab;$$

and since the polygons are similar, we may write the following:

$$AB : ab :: BC : bc$$
$$AB : ab :: CD : cd$$
$$AB : ab :: DE : de, \text{ etc. etc.}$$

Hence, (Th. 7),

$$AB : ab :: AB+BC+CD+DE, \text{ etc.} : ab+bc+cd+de, \text{ etc.}$$

Therefore, the perimeters of similar polygons are to one another as their homologous sides. This is the first part of the theorem.

Since the polygons are similar, the triangles EAB, eab, are similar, and if the triangle EAB is a part expressed by the fraction $\frac{1}{n}$, of the polygon to which it belongs, the triangle eab is a like part of the other polygon.

Therefore, $EAB : eab :: ABCDEA : abcdea.$

But, (Th. 20), $EAB : eab :: \overline{AB}^2 : \overline{ab}^2.$

Therefore, (Th. 6),

$$ABCDEA : abcdea :: \overline{AB}^2 : \overline{ab}^2.$$

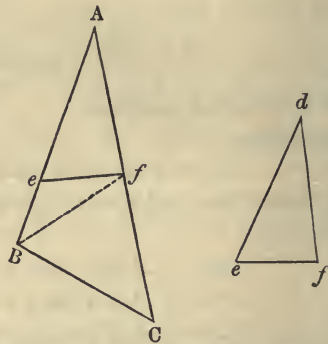
Therefore, the similar polygons are to one another as the squares on their homologous sides. This is the second part of the theorem.

Hence the theorem; *the perimeters of similar polygons are to one another, etc.*

THEOREM XXIII.

Two triangles which have an angle in the one equal to an angle in the other, are to each other as the rectangle of the sides about the equal angles.

Let ABC and def be two triangles having the angles A and d equal. It is to be proved that the areas ABC and def are to each other as $AB.AC$ is to $de.df$.



Conceive the triangle def placed on the triangle ABC, so that d shall fall on A, and de on AB; then df will fall on AC, because the ∟'s A and d are equal. On AB, lay off Ae, equal to de; and on AC, lay off Af, equal to df, and draw ef. The triangle Aef will then be equal to the triangle def. Join B and f.

Now, as triangles having the same altitude are to each other as their bases, (Th. 16, Cor. 1), we have

$$Aef : ABf :: Ae : AB$$

also,

$$ABf : ABC :: Af : AC$$

Multiplying these proportions together, term by term, omitting from the result ABf, a factor common to the terms of the first couplet, we have

$$Aef : ABC :: Ae \,.\, Af : AB \,.\, AC$$

But Aef is equal to def, Ae to de, and Af to df; therefore,

$$def : ABC :: de \,.\, df : AB \,.\, AC$$

Hence the theorem; *two triangles which have an angle, etc.*

Scholium.—If we suppose that

$$AB : AC :: de : df,$$

the two triangles will be similar; and if we multiply the terms of the first couplet of this proportion by AC, and the terms of the second couplet by df, we shall have

$$AB \,.\, AC : \overline{AC}^2 :: de \,.\, df : \overline{df}^2$$

or,

$$AB \,.\, AC : de \,.\, df :: \overline{AC}^2 : \overline{df}^2$$

Comparing this with the last proportion in this theorem, and we have, (Th. 6);

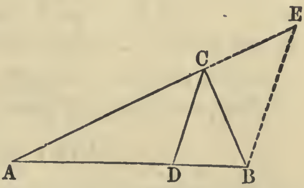
$$def : ABC :: \overline{df}^2 : \overline{AC}^2$$

REMARK. — This scholium is therefore another demonstration of Theorem 20, and hence that theorem need not necessarily have been made a distinct proposition. We require no stronger proof of the certainty of geometrical truth, than the fact that, however different the processes by which we arrive at these truths, we are never led into inconsistencies; but whenever our conclusions can be compared, they will harmonize with each other completely, provided our premises are true and our reasoning logical.

It is hoped that the student will lose no opportunity to exercise his powers, and test his skill and knowledge, in seeking original demonstrations of theorems, and in deducing consequences and conclusions from those already established.

THEOREM XXIV.

If the vertical angle of a triangle be bisected, the bisecting line will cut the base into segments proportional to the adjacent sides of the triangle.



Let ABC be any triangle, and the vertical angle, C, be bisected by the straight line CD. Then we are to prove that

$$AD : DB = AC : CB.$$

Produce AC to E, making $CE = CB$, and draw EB. The exterior angle ACB, of the $\triangle$ CEB, is equal to the two angles E, and CBE; but the angle $E = CBE$, because $CB = CE$, and the triangle is isosceles; therefore the angle ACD, the half of the angle ACB, is equal to the angle E, and DC and BE are parallel, (Cor. 2, Th. 7, B. I).

Now, as ABE is a triangle, and CD is parallel to BE, we have $AD : DB = AC : CE$ or CB, (Th. 17).

Hence the theorem; *if the vertical angle of a triangle be bisected, etc.*

THEOREM XXV.

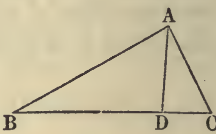
If from the right angle of a right-angled triangle, a perpendicular is drawn to the hypotenuse;

1. *The perpendicular divides the triangle into two similar triangles, each of which is similar to the whole triangle.*

2. *The perpendicular is a mean proportional between the segments of the hypotenuse.*

3. *The segments of the hypotenuse are in proportion to the squares on the adjacent sides of the triangle.*

4. *The sum of the squares on the two sides is equivalent to the square on the hypotenuse.*



Let BAC be a triangle, right angled at A; and draw AD perpendicular to BC.

1. The two △'s, ABC and ABD, have the common angle, B, and the right angle BAC = the right angle BDA; therefore, the third ∟'s are equal, and the two △'s are similar by Th. 17, Cor. 1. In the same manner we prove the △ ADC similar to the △ ABC; and the two triangles, ADB, ADC, being similar to the same △ ABC, are similar to each other.

2. As similar triangles have the sides about the equal angles proportional, (Def. 16), we have

$$BD : AD :: AD : CD;$$

or, *the perpendicular is a mean proportional between the segments of the hypotenuse.*

3. Again, $BC : BA :: BA : BD$
hence, $\overline{BA}^2 = BC.BD$ (1)
also, $BC : CA :: CA : CD$
hence, $\overline{CA}^2 = BC.CD$ (2)

Dividing Eq. (1) by Eq. (2), member by member, we obtain

$$\frac{\overline{BA}^2}{\overline{CA}^2} = \frac{BD}{CD}$$

which, in the form of a proportion, is

$$\overline{CA}^2 : \overline{BA}^2 :: CD : BD;$$

that is, *the segments of the hypotenuse are proportional to the squares on the adjacent sides.*

4. By the addition of (1) and (2), we have

$$\overline{BA}^2 + \overline{CA}^2 = BC(BD + CD) = \overline{BC}^2;$$

that is, *the sum of the squares on the sides about the right angle is equivalent to the square on the hypotenuse.* This is another demonstration of Theorem 39, B. I.

Hence the theorem, *if from the right angle of a right-angled triangle, etc.*

BOOK III.

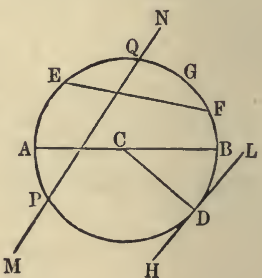
OF THE CIRCLE, AND THE INVESTIGATION OF THEOREMS DEPENDENT ON ITS PROPERTIES.

DEFINITIONS.

1. * A **Curved Line** is one whose consecutive parts, however small, do not lie in the same direction.

2. A **Circle** is a plane figure bounded by one uniformly curved line, all of the points of which are at the same distance from a certain point within, called the *center*.

3. The **Circumference** of a circle is the curved line that bounds it.



4. The **Diameter** of a circle is a line passing through the center, and terminating at both extremities in the circumference. Thus, in the figure, C is the center of the circle, the curved line $AGBD$ is the circumference, and AB is a diameter.

5. The **Radius** of a circle is a line extending from the center to any point in the circumference. Thus, CD is a radius of the circle.

6. An **Arc** of a circle is any portion of the circumference.

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* The first six of the above definitions have been before given among the general definitions of Geometry, but it was deemed advisable to reinsert them here.

7. A **Chord** of a circle is the line connecting the extremities of an arc.

8. A **Segment** of a circle is the portion of the circle on either side of a chord.

Thus, in the last figure, *EGF* is an arc, and *EF* is a chord of the circle, and the spaces bounded by the chord *EF*, and the two arcs *EGF* and *EDF*, into which it divides the circumference, are segments.

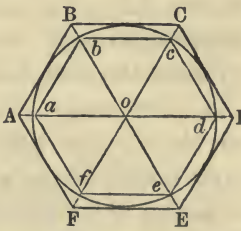
9. A **Tangent** to a circle is a line which, meeting the circumference at any point, will not cut it on being produced. The point in which the tangent meets the circumference is called the *point of tangency.*

10. A **Secant** to a circle is a line which meets the circumference in two points, and lies a part within and a part without the circumference.

11. A **Sector** of a circle is a portion of the circle included between any two radii and their intercepted arc.

Thus, in the last figure, the line *HL*, which meets the circumference at the point *D*, but does not cut it, is a tangent, *D* being the point of tangency; and the line *MN*, which meets the circumference at the points *P* and *Q*, and lies a portion within and a portion without the circle, is a secant. The area bounded by the arc *BD*, and the two radii *CB*, *CD*, is a sector of the circle.

12. A **Circumscribed Polygon** is one all of whose sides are tangent to the circumference of the circle; and conversely, the circle is then said to be *inscribed* in the polygon.



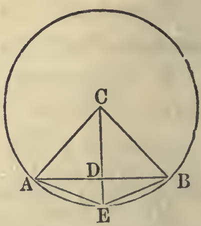
13. An **Inscribed Polygon** is one the vertices of whose angles are all found in the circumference of the circle; and conversely, the circle is then said to be *circumscribed* about the polygon.

14. A **Regular Polygon** is one which is both equiangular and equilateral.

The last three definitions are illustrated by the last figure.

THEOREM I.

Any radius perpendicular to a chord, bisects the chord, and also the arc of the chord.



Let AB be a chord, C the center of the circle, and CE a radius perpendicular to AB; then we are to prove that $AD = BD$, and $AE = EB$.

Since C is the center of the circle, $AC = BC$, CD is common to the two △'s ACD and BCD, and the angles at D are right angles; therefore the two △'s ADC and BDC are equal, and $AD = DB$, which proves the first part of the theorem.

Now, as $AD = DB$, and DE is common to the two spaces, ADE and BDE, and the angles at D are right angles, if we conceive the sector CBE turned over and placed on CAE, CE retaining its position, the point B will fall on the point A, because $AD = BD$ and $AC = BC$; then the arc BE will fall on the arc AE; otherwise there would be points in one or the other arc unequally distant from the center, which is impossible; therefore, the arc $AE =$ the arc EB, which proves the second part of the theorem.

Hence the theorem.

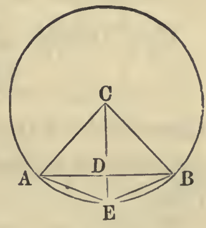
Cor. The center of the circle, the middle point of the chord AB, and of the subtended arc AEB, are three points in the same straight line perpendicular to the chord at its middle point. Now as but one perpendicular can be drawn to a line from a given point in that line, it follows:

1st. That the radius drawn to the middle point of any arc bisects, and is perpendicular to, the chord of the arc.

2d. That the perpendicular to the chord at its middle point passes through the center of the circle and the middle of the subtended arc.

THEOREM II.

Equal angles at the center of a circle are subtended by equal chords.

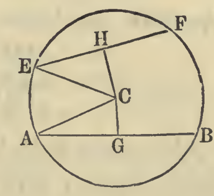


Let the angle $ACE =$ the angle ECB; then the two isosceles triangles, ACE, and ECB, are equal in all respects, and $AE = EB$.

Hence the theorem.

THEOREM III.

In the same circle, or in equal circles, equal chords are equally distant from the center.



Let AB and EF be equal chords, and C the center of the circle. From C, draw CG and CH, perpendicular to the respective chords. These perpendiculars will bisect the chords, (Th. 1), and we shall have $AG = EH$. We are now to prove that $CG = CH$.

Since the △'s ECH and ACG are right-angled, we have, (Th. 39, B. I),

$$\overline{EH}^2 + \overline{HC}^2 = \overline{EC}^2$$

and, $$\overline{AG}^2 + \overline{GC}^2 = \overline{AC}^2.$$

By subtracting these equations, member from member, we find that

$$\overline{EH}^2 - \overline{AG}^2 + \overline{HC}^2 - \overline{GC}^2 = \overline{EC}^2 - \overline{AC}^2 \quad (1)$$

But the chords are equal by hypothesis, hence their halves, EH and AG, are equal; also $EC = AC$, being radii of the circle. Wherefore,

$$\overline{EH}^2 - \overline{AG}^2 = 0$$

and, $$\overline{EC}^2 - \overline{AC}^2 = 0.$$

These values in Equation (1) reduce it to

$$\overline{HC}^2 - \overline{GC}^2 = 0$$

or, $$\overline{HC}^2 = \overline{GC}^2$$

and, $$HC = GC.$$

Hence the theorem.

Cor. Under all circumstances we have

$$\overline{EH}^2 + \overline{HC}^2 = \overline{AG}^2 + \overline{GC}^2,$$

because the sum of the squares in either member of the equation is equivalent to the square of the radius of the circle.

Now, if we suppose HC greater than GC, then will $\overline{HC}^2$ be greater than $\overline{GC}^2$. Let the difference of these squares be represented by d.

Subtracting $\overline{GC}^2$ from both members of the above equation, we have

$$\overline{EH}^2 + d = \overline{AG}^2$$

whence, $\overline{AG}^2 > \overline{EH}^2$, and $AG > EH$.

Therefore, AB, the double of AG, is greater than EF, the double of EH; that is, *of two chords in the same or equal circles, the one nearer the center is the greater.*

The equation, $\overline{EH}^2 + \overline{HC}^2 = \overline{AG}^2 + \overline{GC}^2$, being true, whatever be the position of the chords, we may suppose GC to have any value between 0 and AC, the radius of the circle.

When GC becomes zero, the equation reduces to

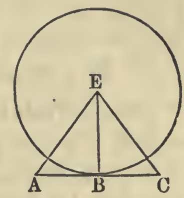
$$\overline{EH}^2 + \overline{HC}^2 = \overline{AG}^2 = R^2;$$

that is, under this supposition, AG coincides with AC, and AB becomes the diameter of the circle, *the greatest chord that can be drawn in it.*

THEOREM IV

A line tangent to the circumference of a circle is at right angles with the radius drawn to the point of contact.

Let AC be a line tangent to the circle at the point B, and draw the radius, EB, and the lines, AE and CE.



Now, we are to prove that EB is perpendicular to AC. Because B is the only point in the line AC which meets the circle, (Def. 9, B. III), any other line, as AE or CE, must be greater than EB; therefore, EB is the shortest line that can be drawn from the point E to the line AC; and EB is the perpendicular to AC, (Th. 23, B. I).

Hence the theorem.

THEOREM V.

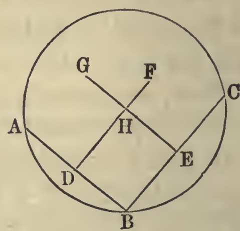
In the same circle, or in equal circles, equal chords subtend or stand on equal portions of the circumference.

Conceive two equal circles, and two equal chords drawn within them. Then, conceive one circle taken up and placed upon the other, center upon center, in such a position that the two equal chords will fall on, and exactly coincide with, each other; the circles must also coincide, because they are equal; and the two arcs of the two circles on either side of the equal chords must also coincide, or the circles could not coincide; and magnitudes which coincide, or exactly fill the same space, are in all respects equal, (Ax. 10).

Hence the theorem.

THEOREM VI.

Through three given points, not in the same straight line, one circumference can be made to pass, and but one.



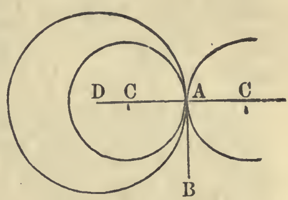
Let A, B, and C be three given points, not in the same straight line, and draw the lines AB and BC. If a circumference is made to pass through the two points A and B, the line AB will be a chord to such a circle; and if a chord is bisected by a line at right angles, the bisecting line will pass through the center of the circle, (Cor., Th. 1); therefore, if we bisect the line AB, and draw DF, perpendicular to AB, at the point of bisection, any circumference that can pass through the points, A and B, must have its center somewhere in the line DF. And if we draw EG at right angles to BC at its middle point, any circumference that can pass through the points B and C must have its center somewhere in the line EG. Now, if the two lines, DF and EG, meet in a common point, that point will be a center, about which a circumference can be drawn to pass through the three points, A, B, and C, and DF and EG will meet in every case, unless they are parallel; but they are not parallel, for if they were, it would follow (Th. 5, B. I) that, since DF is intersected at right angles by the line AB, it must also be intersected at right angles by the line BC, having a direction different from that of AB; which is impossible, (Th. 7, B. I).

Therefore the two lines will meet; and, with the point H, at which they meet, as a center, and $HB = HA = HC$ as a radius, one circumference, and but one, can be made to pass through the three given points.

Hence the theorem.

THEOREM VII.

If two circles touch each other, either internally or externally, the two centers and the point of contact will be in one right line.



Let two circles touch each other internally, as represented at A, and conceive AB to be a tangent at the common point A. Now, if a line, perpendicular to AB, be drawn from the point A, it must pass through the center of each circle, (Th. 4); and as but one perpendicular can be drawn to a line at a given point in it, A, C, and D, the point of contact and the two centers must be in one and the same line.

Next, let two circles touch each other externally, and from the point of contact conceive the common tangent, AB, to be drawn.

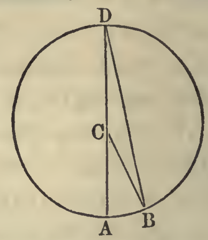
Then a line, AC, perpendicular to AB, will pass through the center of one circle, (Th. 4), and a perpendicular, AD, from the same point, A, will pass through the center of the other circle; hence, BAC and BAD are together equal to two right angles; therefore CAD is one continued straight line, (Th. 3, B. I).

Cor. When two circles touch each other internally, the distance between their centers is equal to the difference of their radii; and when they touch each other externally, the distance between their centers is equal to the sum of their radii.

THEOREM VIII.

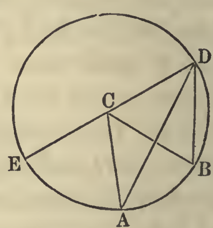
An angle at the circumference of any circle is measured by one half the arc on which it stands.

In this work it is taken as an axiom that any angle whose vertex is at the center of a circle, is measured by the arc on which it stands; and we now proceed to prove that when the arcs are equal, the angle at the circumference is equal to one half the angle at the center.



Let ACB be an angle at the center, and D an angle at the circumference, and at first suppose D in a line with AC. We are now to prove that the angle ACB is double the angle D.

The $\triangle\, DCB$ is an isosceles triangle, because $CD = CB$; and its exterior angle, ACB, is equal to the two interior angles, D, and CBD, (Th. 12, B. I), and since these two angles are equal to each other, the angle ACB is double the angle at D. But ACB is measured by the arc AB; therefore the angle D is measured by one half the arc AB.



Next, suppose D not in a line with AC, but at any point in the circumference, except on AB; produce DC to E.

Now, by the first part of this theorem,

the angle $ECB = 2EDB$,
also, $ECA = 2EDA$,
by subtraction, $ACB = 2ADB$.

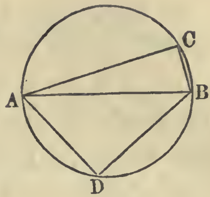
But ACB is measured by the arc AB; therefore ADB or the angle D, is measured by one half of the same arc Hence the theorem.

THEOREM IX.

An angle in a semicircle is a right angle; an angle in a segment greater than a semicircle is less than a right angle; and an angle in a segment less than a semicircle is greater than a right angle.

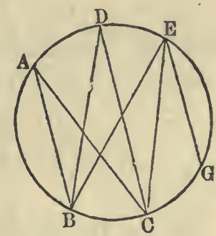
If the angle ACB is in a semicircle, the opposite segment, ADB, on which it stands, is also a semicircle; and the angle ACB is measured by one half the arc ADB

(Th. 8); that is, one half of 180°, or 90°, which is the measure of a right angle.



If the angle ACB is in a segment greater than a semicircle, then the opposite segment is less than a semicircle, and the measure of the angle is less than one half of 180°, or less than a right angle. If the angle ACB is in a segment less than a semicircle, then the opposite segment, ADB, on which the angle stands, is greater than a semicircle, and its half is greater than 90°; and, consequently, the angle is greater than a right angle.

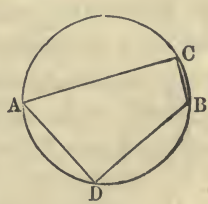
Hence the theorem.



Cor. Angles at the circumference, and standing on the same arc of a circle, are equal to one another; for all angles, as BAC, BDC, BEC, are equal, because each is measured by one half of the arc BC. Also, if the angle BEC is equal to CEG, then the arcs BC and CG are equal, because their halves are the measures of equal angles.

THEOREM X.

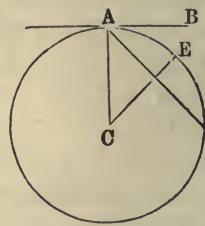
The sum of two opposite angles of any quadrilateral inscribed in a circle, is equal to two right angles.



Let $ACBD$ represent any quadrilateral inscribed in a circle. The angle ACB has for its measure, one half of the arc ADB, and the angle ADB has for its measure, one half of the arc ACB; therefore, by addition, the sum of the two opposite angles at C and D, are together measured by one half of the whole circumference, or by 180 degrees, = two right angles. Hence the theorem.

THEOREM XI.

An angle formed by a tangent and a chord is measured by one half of the intercepted arc.



Let AB be a tangent, and AD a chord, and A the point of contact; then we are to prove that the angle BAD is measured by one half of the arc AED.

From A draw the radius AC; and from the center, C, draw CE perpendicular to AD.

The $\angle BAD + \angle DAC = 90°$, (Th. 4).

Also, $\angle C + \angle DAC = 90°$, (Cor. 4, Th. 12, B. 1).

Therefore, by subtraction, $BAD - C = 0$;

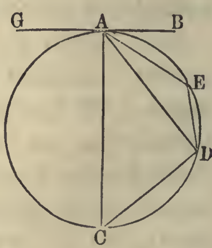
by transposition, the angle $BAD = C$.

But the angle C, at the center of the circle, is measured by the arc AE, the half of AED; therefore, the equal angle, BAD, is also measured by the arc AE, the half of AED.

Hence the theorem.

THEOREM XII.

An angle formed by a tangent and a chord, is equal to an angle in the opposite segment of the circle.



Let AB be a tangent, and AD a chord, and from the point of contact, A, draw any angles, as ACD, and AED, in the segments. Then we are to prove that $\angle BAD = \angle ACD$, and $\angle GAD = \angle AED$.

By Th. 11, the angle BAD is measured by one half the arc AED; and as the angle ACD is measured by one half of the same arc, (Th. 8), we have $\angle BAD = \angle ACD$.

Again, as $AEDC$ is a quadrilateral, inscribed in a circle, the sum of the opposite angles,

$$ACD + AED = 2 \text{ right angles.} \quad \text{(Th. 10).}$$

Also, the sum of the angles

$$BAD + DAG = 2 \text{ right angles.} \quad \text{(Th. 1, B. I).}$$

By subtraction (and observing that BAD has just been proved equal to ACD), we have,

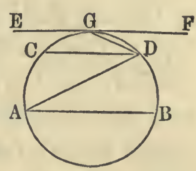
$$AED - DAG = 0.$$

Or, by transposition, $\quad AED = DAG.$

Hence the theorem.

THEOREM XIII.

Arcs of the circumference of a circle intercepted by parallel chords, or by a tangent and a parallel chord, are equal.



Let AB and CD be parallel chords, and draw the diagonal, AD; now, because AB and CD are parallel, the angle DAB = the angle ADC (Th. 6, B. I); but the angle DAB has for its measure, one half of the arc BD; and the angle ADC has for its measure, one half of the arc AC, (Th. 8); and because the angles are equal, the arcs are equal; that is, the arc BD = the arc AC.

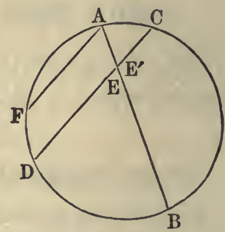
Next, let EF be a tangent, parallel to a chord, CD, and from the point of contact, G, draw GD.

Since EF and CD are parallel, the angle CDG = the angle DGF. But the angle CDG has for its measure, one-half of the arc CG, (Th. 8); and the angle DGF has for its measure, one half of the arc GD, (Th. 11); therefore, these measures of equals must be equal; that is, the arc CG = the arc GD.

Hence, the theorem.

THEOREM XIV.

When two chords intersect each other within a circle, the angle thus formed is measured by one half the sum of the two intercepted arcs.



Let AB and CD intersect each other within the circle, forming the two angles, E and E', with their equal vertical angles.

Then, we are to prove that the angle E is measured by one half the sum of the arcs AC and BD; and the angle E' is measured by one half the sum of the arcs AD and CB.

First, draw AF parallel to CD, and FD will be equal to AC, (Th. 13); then, by reason of the parallels, $\angle BAF = \angle E$. But the angle BAF is measured by one half of the arc BDF; that is, one half of the arc BD plus one half of the arc AC.

Now, as the sum of the angles E and E' is equal to two right angles, that sum is measured by one half the whole circumference.

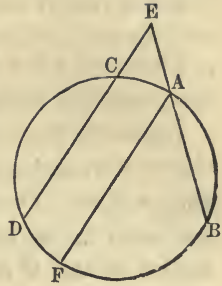
But the angle E, alone, as we have just proved, is measured by one half the sum of the arcs BD and AC; therefore, the other angle, E', is measured by one half the sum of the other parts of the circumference,

$$AD + CB.$$

Hence the theorem.

THEOREM XV.

When two secants intersect, or meet each other without a circle, the angle thus formed is measured by one half the difference of the intercepted arcs.

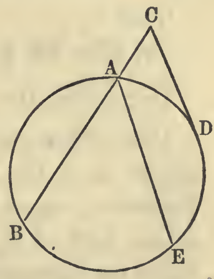


Let DE and BE be two secants meeting at E; and draw AF parallel to CD. Then, by reason of the parallels, the angle E, made by the intersection of the two secants, is equal to the angle BAF. But the angle BAF is measured by one half the arc BF; that is, by one half the difference between the arcs BD and AC.

Hence the theorem.

THEOREM XVI.

The angle formed by a secant and a tangent is measured by one half the difference of the intercepted arcs.



Let BC be a secant, and CD a tangent, meeting at C. We are to prove that the angle formed at C, is measured by one half the difference of the arcs BD and DA.

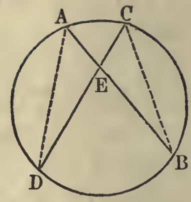
From A, draw AE parallel to CD; then the arc AD = the arc DE; $BD - DE = BE$; and the $\angle BAE = \angle C$. But the angle BAE is measured by one half the arc BE, (Th. 8,) that is, by one half the difference between the arcs BD and AD; therefore, the equal angle, C, is measured by one half the arc BE.

Hence the theorem.

THEOREM XVII.

When two chords intersect each other in a circle, the rectangle contained by the segments of the one, will be equivalent to the rectangle contained by the segments of the other.

Let AB and CD be two chords intersecting each other in E. Then we are to prove that the rectangle $AE \times EB =$ the rectangle $CE \times ED$.



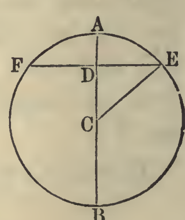
Draw the lines AD and CB, forming the two triangles AED and CEB. The angles B and D are equal, because they are each measured by one half the arc, AC. Also the angles A and C are equal, because each is measured by one half the arc, DB; and $\angle AED = \angle CEB$, because they are vertical angles; hence, the triangles, AED and CEB, are equiangular and similar. But equiangular triangles have their sides about the equal angles proportional, (Cor. 1, Th. 17, B. II); therefore, AE and ED, about the angle E, are proportional to CE and EB, about the same or equal angle.

That is, $\quad AE : ED :: CE : EB$;

Or, (Th. 19, B. II), $\quad AE \times EB = CE \times ED$.

Hence the theorem.

Cor. *When one chord is a diameter, and the other at right angles to it, the rectangle contained by the segments of the diameter is equal to the square of one half the other chord; or one half of the bisected chord is a mean proportional between the segments of the diameter.*



For, $AD \times DB = FD \times DE$. But, if AB passes through the center, C, at right angles to FE, then $FD = DE$ (Th. 1); and in the place of FD, write its equal, DE, in the last equation, and we have

$$AD \times DB = \overline{DE}^2,$$

or, (Th. 3, B II), $\quad AD : DE :: DE : DB$.

Put, $DE = x$, $CD = y$, and $CE = R$, the radius of the circle.

Then $AD = R - y$, and $DB = R + y$. With this notation,

$$AD \times DB = DE^2$$

becomes, $$(R - y)(R + y) = x^2$$

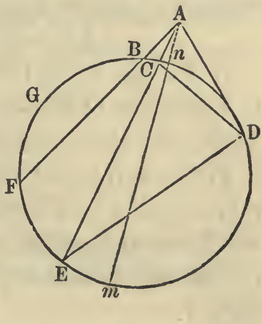
or, $$R^2 - y^2 = x^2$$

or, $$R^2 = x^2 + y^2$$

That is, *the square of the hypotenuse of the right-angled triangle,* DCE, *is equal to the sum of the squares of the other two sides.*

THEOREM XVIII.

If from a point without a circle, a tangent line be drawn to the circumference, and also any secant line terminating in the concave arc, the square of the tangent will be equivalent to the rectangle contained by the whole secant and its external segment.



Let A be a point without the circle DEG, and let AD be a tangent and AE any secant line.

Then we are to prove that

$$AC \times AE = \overline{AD}^2.$$

In the two triangles, ADE and ADC, the angles ADC and AED are equal, since each is measured by one half of the same arc, DC; the angle A is common to the two triangles; their third angles are therefore equal, and the triangles are equiangular and similar.

Their homologous sides give the proportion

$$AE : AD :: AD : AC$$

whence, $$AE \times AC = \overline{AD}^2$$

Hence the theorem.

Cor. If AE and AF are two secant lines drawn from the same point without the circumference, we shall have

$$AC \times AE = \overline{AD}^2$$

and, $$AB \times AF = \overline{AD}^2$$

hence, $$AC \times AE = AB \times AF,$$

which, in the form of a proportion, gives

$$AC : AF :: AB : AE.$$

That is, *the secants are reciprocally proportional to their external segments.*

SCHOLIUM. — By means of this theorem we can determine the diameter of a circle, when we know the length of a tangent drawn from a point without, and the external segment of the secant, which, drawn from the same point, passes through the center of the circle.

Let Am be a secant passing through the center, and suppose the tangent AD to be 20, and the external segment, An, of the secant to be 2. Then, if D denote the diameter, we shall have

$$Am = 2 + D,$$

whence, $$Am \times An = 2\,(2 + D) = 4 + 2D = (20)^2 = 400,$$

$$2D = 396, \text{ and } D = 198.$$

If An, the height of a mountain on the earth, and AD, the distance of the visible sea horizon, be given, we may determine the diameter of the earth.

For example; the perpendicular height of a mountain on the island of Teneriffe is about 3 miles, and its summit can be seen from ships when they are known to be 154 or 155 miles distant; what then is the diameter of the earth?

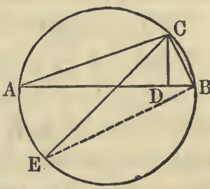
Designate, as before, the diameter by D. Then $Am = 3 + D$, and $Am \times An = 9 + 3D$. $AD = 154.5$; hence, $9 + 3D = (154.5)^2 = 23870.25$, from which we find $D = 7953.75$, which differs but little from the true diameter of the earth.

One source of error, in this mode of computing the diameter of the earth, is atmospheric refraction, the explanation of which does not belong here.

THEOREM XIX.

If a circle be described about a triangle, the rectangle contained by two sides of the triangle is equivalent to the rectangle contained by the perpendicular let fall on the third side, and the diameter of the circumscribing circle.

Let ABC be a triangle, AC and CB, the sides, CD the perpendicular let fall on the base AB, and CE the diameter of the circumscribing circle. Then we are to prove that



$$AC \times CB = CE \times CD.$$

The two $\triangle$'s, ACD and CEB, are equiangular, because $\angle A = \angle E$, both being measured by the half of the arc CB; also, ADC is a right angle, and is equal to CBE, an angle in a semicircle, and therefore a right angle; hence, the third angle, $ACD = \angle BCE$, (Th. 12, Cor. 2, B. I). Therefore, (Cor. 1, Th. 17, B. II),

$$AC : CD :: CE : CB$$

and, $$AC \times BC = CE \times CD.$$

Hence the theorem; *if a circle, etc.*

Cor. The continued product of three sides of a triangle is equal to twice the area of the triangle into the diameter of its circumscribing circle.

Multiplying both members of the last equation by AB, we have,

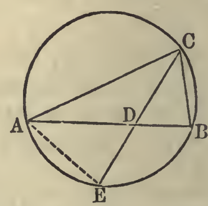
$$AC \times BC \times AB = CE \times (AB \times CD).$$

But CE is the diameter of the circle, and $(AB \times CD)$ = twice the area of the triangle;

Therefore, $AC \times CB \times AB =$ diameter multiplied by twice the area of the triangle.

THEOREM XX.

The square of a line bisecting any angle of a triangle, together with the rectangle of the segments into which it cuts the opposite side, is equivalent to the rectangle of the two sides including the bisected angle.



Let ABC be a triangle, and CD a line bisecting the angle C. Then we are to prove that

$$CD^2 + (AD \times DB) = AC \times CB.$$

The two $\triangle$'s, ACE and CDB, are equiangular, because the angles E and B are equal, both being in the same segment, and the $\angle\ ACE = BCD$, by hypothesis. Therefore, (Th. 17, Cor. 1, B. II),

$$AC : CE :: CD : CB.$$

But it is obvious that $CE = CD + DE$, and by substituting this value of CE, in the proportion, we have,

$$AC : CD + DE :: CD : CB.$$

By multiplying extremes and means,

$$\overline{CD}^2 + (DE \times CD) = AC \times CB.$$

But by (Th. 17),

$$DE \times CD = AD \times DB,$$

and substituting, we have,

$$\overline{CD}^2 + (AD \times DB) = AC \times CB.$$

Hence the theorem.

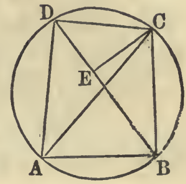
THEOREM XXI.

The rectangle contained by the two diagonals of any quadrilateral inscribed in a circle, is equivalent to the sum of the two rectangles contained by the opposite sides of the quadrilateral.

Let $ABCD$ be a quadrilateral inscribed in a circle; then we are to prove that

$$AC \times BD = (AB \times DC) + (AD \times BC).$$

From C, draw CE, making the angle DCE equal to the angle ACB; and as the angle BAC is equal to the angle CDE, both being in the same segment, therefore, the two triangles, DEC and ABC, are equiangular, and we have (Th. 17, Cor. 1, B. II),



$$AB : AC :: DE : DC \quad (1)$$

The two $\triangle$'s, ADC and BEC, are equiangular; for the $\angle DAC = \angle EBC$, both being in the same segment; and the $\angle DCA = \angle ECB$, for $DCE = BCA$; to each of these add the angle ECA, and $DCA = ECB$; therefore, (Th. 17, Cor. 1, B. II),

$$AD : AC :: BE : BC \quad (2).$$

By multiplying the extremes and means in proportions (1) and (2), and adding the resulting equations, we have,

$$(AB \times DC) + (AD \times BC) = (DE + BE) \times AC.$$

But, $\quad DE + BE = BD$; therefore,

$$(AB \times DC) + (AD \times BC) = AC \times BD.$$

Cor. When two adjacent sides of the quadrilateral are equal, as AB and BC, then the resulting equation is,

$$(AB \times DC) + (AB \times AD) = AC \times BD;$$

or, $$AB \times (DC + AD) = AC \times BD;$$

or, $$AB : AC :: BD : DC + AD.$$

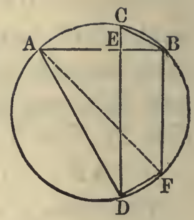
That is, *one of the two equal sides of the quadrilateral is to the adjoining diagonal, as the transverse diagonal is to the sum of the two unequal sides.*

THEOREM XXII.

If two chords intersect each other at right angles in a circle, the sum of the squares of the four segments thus formed is equivalent to the square of the diameter of the circle.

Let AB and CD be two chords, intersecting each other at right angles. Draw BF parallel to ED, and draw DF and AF. Now, we are to prove that

$$\overline{AE}^2 + \overline{EB}^2 + \overline{EC}^2 + \overline{ED}^2 = \overline{AF}^2.$$



As BF is parallel to ED, ABF is a right angle, and therefore AF is a diameter, (Th. 9). Also, because BF is parallel to CD, $CB = DF$, (Th. 13).

Because CEB is a right angle,

$$\overline{CE}^2 + \overline{EB}^2 = \overline{CB}^2 = \overline{DF}^2.$$

Because AED is a right angle,

$$\overline{AE}^2 + \overline{ED}^2 = \overline{AD}^2.$$

Adding these two equations, we have,

$$\overline{CE}^2 + \overline{EB}^2 + \overline{AE}^2 + \overline{ED}^2 = \overline{DF}^2 + \overline{AD}^2.$$

But, as AF is a diameter, and ADF a right angle, (Th. 9),

$$\overline{DF}^2 + \overline{AD}^2 = \overline{AF}^2;$$

therefore, $\overline{CE}^2 + \overline{EB}^2 + \overline{AE}^2 + \overline{ED}^2 = \overline{AF}^2.$

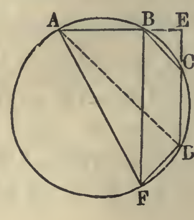
Hence the theorem.

Scholium. — If two chords intersect each other at right angles, in a circle, and their opposite extremities be joined, the two chords thus formed may make two sides of a right-angled triangle, of which the diameter of the circle is the hypotenuse.

For, AD is one of these chords, and CB is the other; and we have shown that $CB = DF$; and AD and DF are two sides of a right-angled triangle, of which AF is the hypotenuse; therefore, AD and CB may be considered the two sides of a right-angled triangle, and AF its hypotenuse.

THEOREM XXIII.

If two secants intersect each other at right angles, the sum of their squares, increased by the sum of the squares of the two segments without the circle, will be equivalent to the square of the diameter of the circle.



Let AE and ED be two secants intersecting at right angles at the point E. From B, draw BF parallel to CD, and draw AF and AD. Now we are to prove that

$$\overline{EA}^2 + \overline{ED}^2 + \overline{EB}^2 + \overline{EC}^2 = \overline{AF}^2.$$

Because BF is parallel to CD, ABF is a right angle, and consequently AF is a diameter, and $BC=DF$; and because AF is a diameter, ADF is a right angle. As AED is a right angle,

$$\overline{AE}^2+\overline{ED}^2=\overline{AD}^2$$

Also, $$\overline{EB}^2+\overline{EC}^2=\overline{BC}^2=\overline{DF}^2$$

By addition, $$\overline{AE}^2+\overline{ED}^2+\overline{EB}^2+\overline{EC}^2=\overline{AD}^2+\overline{DF}^2=\overline{AF}^2$$

Hence the theorem.

THEOREM XXIV.

If perpendiculars be drawn bisecting the three sides of a triangle, they will, when sufficiently produced, meet in a common point.

The three angular points of a triangle are not in the same straight line; consequently one circumference, and but one, may be made to pass through them.

Conceive a triangle to be thus circumscribed. The sides of the triangle then become chords of the circumscribing circle. Now if these sides be bisected, and at the points of bisection perpendiculars be drawn to the sides, each of these perpendiculars will pass through the center of the circle (Th. 1, Cor.); and the perpendiculars will therefore meet in a common point.

Hence the theorem.

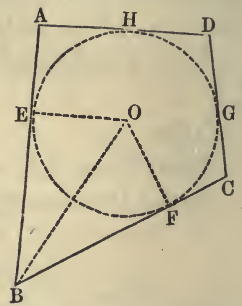
THEOREM XXV.

The sums of the opposite sides of a quadrilateral circumscribing a circle are equal.

Let $ABCD$ be a quadrilateral circumscribed about a circle, whose center is O. Then we are to prove that

$$AB+DC=AD+BC.$$

From the center of the circle draw OE and OF to the points of contact of the sides AB and BC. Then, the two right-angled triangles, OEB and OFB, are equal, because they have the hypotenuse OB common, and the side $OF = OE$; therefore, $BE = BF$, (Cor., Th. 39, B. I).



In like manner we can prove that

$AE = AH$, $CF = CG$, and $DG = DH$.

Now, taking the equation $BE = BF$, and adding to its first member CG, and to its second the equal line CF. we have,

$$BE + CG = BF + CF \quad (1)$$

The equation $AE = AH$, by adding to its first member DG, and to the second the equal line, DH, gives

$$AE + DG = AH + DH \quad (2)$$

By the addition of (1) and (2), we find that

$$BE + AE + CG + DG = BF + CF + AH + DH.$$

That is, $$AB + CD = BC + AD.$$

Hence the theorem.

BOOK IV.

PROBLEMS

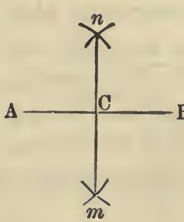
In this section, we have, in most instances, merely shown the construction of the problem, and referred to the theorem or theorems that the student may use, to prove that the object is attained by the construction.

In obscure and difficult problems, however, we have gone through the demonstration as though it were a theorem.

PROBLEM I.

To bisect a given finite straight line.

Let AB be the given line, and from its extremities, A and B, with any radius greater than one half of AB, (Postulate 3), describe arcs, cutting each other in n and m. Draw the line nm; and C, where it cuts AB, will be the middle of the given line.

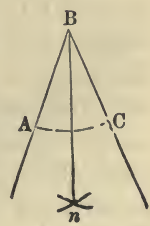


Proof, (B. I, Th. 18, Sch. 2).

PROBLEM II.

To bisect a given angle.

Let ABC be the given angle. With any radius, and B as a center, describe the arc AC. From A and C, as centers, with a radius greater than one half of AC, describe arcs, intersecting in n; join B and n; the joining line will bisect the given angle.

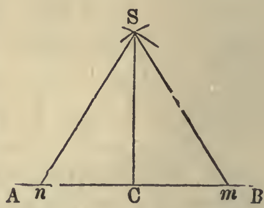


Proof, (Th. 21, B. I).

PROBLEM III.

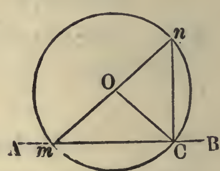
From a given point in a given line, to draw a perpendicular to that line.

Let AB be the given line, and C the given point. Take n and m, at equal distances on opposite sides of C; and with the points m and n, as centers, and any radius greater than nC or mC, describe arcs cutting each other in S. Draw



SC, and it will be the perpendicular required. Proof, (B. I, Th. 18, Sch. 2).

The following is another method, which is preferable, when the given point, C, is at or near the end of the line.

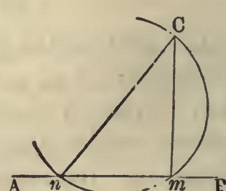


Take any point, O, which is manifestly one side of the perpendicular, as a center, and with OC as a radius, describe a circumference, cutting AB in m and C. Draw mn through the points m and O, and meeting the arc again in n; mn is then a diameter to the circle. Draw Cn, and it will be the perpendicular required. Proof, (Th. 9, B. III).

PROBLEM IV.

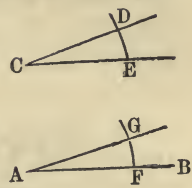
From a given point without a line, to draw a perpendicular to that line.

Let AB be the given line, and C the given point. From C draw any oblique line, as Cn. Find the middle point of Cn by Problem 1, and with that point, as a center, describe a semicircle, having Cn as a diameter. From m, where this semi-circumference cuts AB, draw Cm, and it will be the perpendicular required. Proof, (Th. 9, B. III).



PROBLEM V.

At a given point in a line, to construct an angle equal to a given angle.



Let A be the point given in the line AB, and DCE the given angle.

With C as a center, and any radius, CE, draw the arc ED.

With A as a center, and the radius $AF = CE$, describe an indefinite arc; and with F as a center, and FG as a radius, equal to ED, describe an arc, cutting the other arc in G, and draw AG; GAF will be the angle required. Proof, (Th. 2, B. III).

PROBLEM VI.

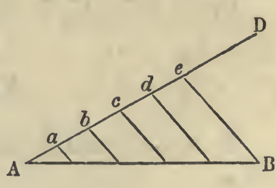
From a given point, to draw a line parallel to a given line.

Let A be the given point, and BC the given line. Draw AC, making an angle, ACB; and from the given point, A, in the line AC, draw the angle $CAD = ACB$, by Problem 5.

Since AD and BC make the same angle with AC, they are, therefore, parallel, (B. I, Th. 7, Cor. 1).

PROBLEM VII.

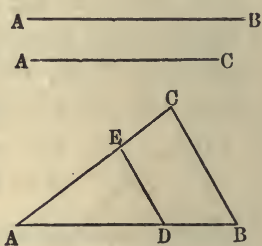
To divide a given line into any number of equal parts.



Let AB represent the given line, and let it be required to divide it into any number of equal parts, say five. From one end of the line A, draw AD, indefinite in both length and position. Take any convenient distance in the dividers, as Aa, and set it off on the line AD, thus making the parts Aa, ab, bc, etc., equal. Through the last point, e, draw EB, and through the points a, b, c, and d, draw parallels to eB, by Problem 6; these parallels will divide the line as required. Proof, (Th. 17, Book II).

PROBLEM VIII.

To find a third proportional to two given lines.

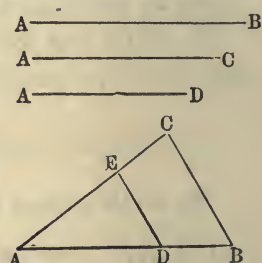


Let AB and AC be any two lines. Place them at any angle, and draw CB. On the greater line, AB, take $AD = AC$, and through D, draw DE parallel to BC; AE is the third proportional required.

Proof, (Th. 17, B. II).

PROBLEM IX.

To find a fourth proportional to three given lines.

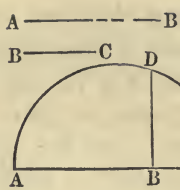


Let AB, AC, AD, represent the three given lines. Place the first two at any angle, as BAC, and draw BC. On AB place AD, and from the point D, draw DE parallel to BC, by Problem 6; AE will be the fourth proportional required.

Proof, (Th. 17, B. II).

PROBLEM X.

To find the middle, or mean proportional, between two given lines.

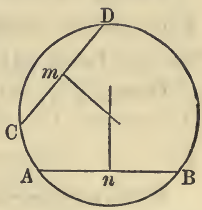


Place AB and BC in one right line, and on AC, as a diameter, describe a semicircle, (Postulate 3), and from the point B, draw BD at right angles to AC, (Problem 3); BD is the mean proportional required.

Proof, (B. III, Th. 17, Cor.).

PROBLEM XI.

To find the center of a given circle.

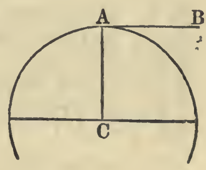


Draw any two chords in the given circle, as AB and CD, and from the middle points, m and n, draw perpendiculars to AB and CD; the point at which these two perpendiculars intersect will be the center of the circle.

Proof, (B. III, Th. 1, Cor.).

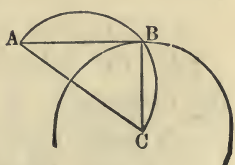
PROBLEM XII.

To draw a tangent to a given circle, from a given point, either in or without the circumference of the circle.



When the given point is in the circumference, as A, draw the radius AC, and from the point A, draw AB perpendicular to AC; AB is the tangent required.

Proof, (Th. 4, B. III).

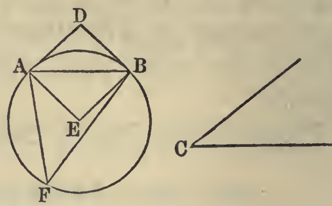


When the given point is without the circle, as A, draw AC to the center of the circle; on AC, as a diameter, describe a semicircle; and from B, where the semi-circumference cuts the given circumference, draw AB, and it will be tangent to the circle.

Proof, (Th. 9, B. III), and, (Th. 4, B. III).

PROBLEM XIII.

On a given line, to describe a segment of a circle, that shall contain an angle equal to a given angle.

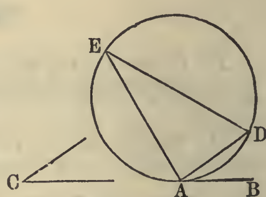


Let AB be the given line, and C the given angle. At the ends of the given line, form angles DAB, DBA, each equal to the given angle, C. Then draw AE and BE perpendiculars to AD and BD; and with E as a center, and EA, or EB, as a radius, describe a circle; then AFB will be the segment required, as any angle F, made in it, will be equal to the given angle, C.

Proof, (Th. 11, B. III), and (Th. 8, B. III).

PROBLEM XIV.

From any given circle to cut a segment, that shall contain a given angle.

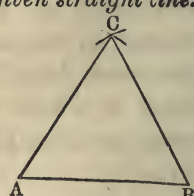


Let C be the given angle. Take any point, as A, in the circumference, and from that point draw the tangent AB; and from the point A, in the line AB, construct the angle $BAD = C$, (Problem 5), and AED is the segment required.

Proof, (Th. 11, B. III), and (Th. 8, B. III).

PROBLEM XV.

To construct an equilateral triangle on a given straight line.

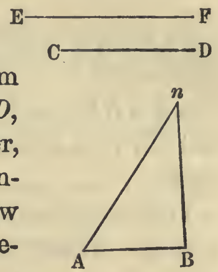


Let AB be the given line; from the extremities A and B, as centers, with a radius equal to AB, describe arcs cutting each other at C. From C, the point of intersection, draw CA and CB; ABC will be the triangle required.

The construction is a sufficient demonstration. Or, (Ax. 1).

PROBLEM XVI.

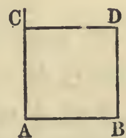
To construct a triangle, having its three sides equal to three given lines, any two of which shall be greater than the third.



Let AB, CD, and EF, represent the three lines. Take any one of them, as AB, to be one side of the triangle. From B, as a center, with a radius equal to CD, describe an arc; and from A, as a center, with a radius equal to EF, describe another arc, cutting the former in n. Draw An and Bn, and AnB will be the $\triangle$ required. Proof, (Ax. 1).

PROBLEM XVII.

To describe a square on a given line.



Let AB be the given line; and from the extremities, A and B, draw AC and BD perpendicular to AB. (Problem 3.)

From A, as a center, with AB as radius, strike an arc across the perpendicular at C; and from C draw CD parallel to AB; $ACDB$ is the square required. Proof, (Th. 26, B. I).

PROBLEM XVIII.

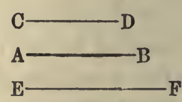
To construct a rectangle, or a parallelogram, whose adjacent sides are equal to two given lines.

Let AB and AC be the two given lines. From the extremities of one line, draw perpendiculars to that line, as in the last problem; and from these perpendiculars, cut off portions equal to the other line; and, by a parallel, complete the figure.

When the figure is to be a parallelogram, with oblique angles, describe the angles by Problem 5. Proof, (Th 26, B. I).

PROBLEM XIX.

To describe a rectangle that shall be equivalent to a given square, and have a side equal to a given line.



Let AB be a side of the given square, and CD one side of the required rectangle.

Find the third proportional, EF, to CD and AB, (Problem 8). Then we shall have

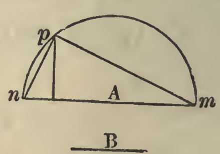
$$CD : AB :: AB : EF.$$

Construct a rectangle with the two given lines, CD and EF, (Problem 18), and it will be equal to the given square, (Th. 3, B. II).

PROBLEM XX.

To construct a square that shall be equivalent to the difference of two given squares.

Let A represent a side of the greater of two given squares, and B a side of the less square.

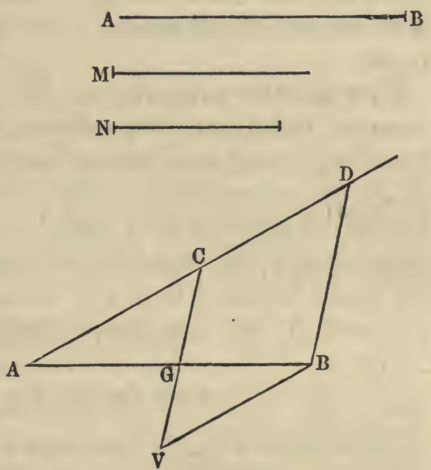


On A, as a diameter, describe a semicircle, and from one extremity, n, as a center, with a radius equal to B, describe an arc, and, from the point where it cuts the circumference, draw mp and np; mp is the side of a square, which, when constructed, (Problem 17), will be equal to the difference of the two given squares. Proof (Th. 9, B. III, and Th. 39, B. I.)

To construct a square equivalent to the sum of two given squares, we have only to draw through any point two lines at right angles, and lay off on one a distance equal to the side of one of the squares, and on the other a distance equal to the side of the other. The straight line connecting the extremities of these lines will be the side of the required square, (Th. 39, B. I).

PROBLEM XXI.

To divide a given line into two parts, which shall be in the ratio of two other given lines.



Let AB be the line to be divided, and M and N the lines having the ratio of the required parts of AB. From the extremity A draw AD, making any angle with AB, and take $AC = M$, and $CD = N$. Join the points D and B by a straight line, and through C draw CG parallel to BD. Then will the point G divide the line AB into parts having the required ratio. (Proof, Th. 17, B. II).

Or, having drawn AD, lay off $AC = M$, and through B draw BV parallel to AD, making it equal to N, and join C and V by a line cutting AB in the point G.

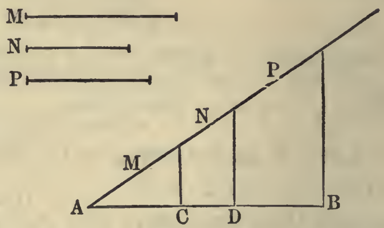
Then the two triangles ACG and GBV are equiangular and similar, and their homologous sides give the proportion,

$$AG : GB :: AC : BV :: M : N$$

The line AB is therefore divided, at the point G, into parts which are in the ratio of the lines M and N.

PROBLEM XXII.

To divide a given line into any number of parts, having to each other the ratios of other given lines.

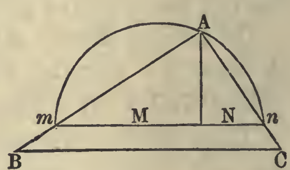


Let AB be the given line to be divided, and M, N, P, etc., the lines to which the parts of AB are to be proportional.

Through the point A draw an indefinite line, making, with AB, any convenient angle, and on this line lay off from A the lines M, N, P, etc., successively. Join the extremity of the last line to the point B by a straight line, parallel to which draw other lines through the points of division of the indefinite line, and they will divide the line AB at the points C, D, etc., into the required parts. (Proof, Th. 17, B. II).

PROBLEM XXIII.

To construct a square that shall be to a given square, as a line, M, *to a line,* N.



Place M and N in a line, and on the sum describe a semicircle. From the point where the two lines meet, draw a perpendicular to meet the circumference in A. Draw Am and An, and produce them indefinitely. On An or An produced, take $AC =$ to the side of the given square ; and from C, draw CB parallel to mn ; AB is a side of the required square.

For, $\overline{Am}^2 : \overline{An}^2 :: \overline{AB}^2 : \overline{AC}^2$, (Th. 17, B. II).
Also, $\overline{Am}^2 : \overline{An}^2 :: M : N$, (Th. 25, B. II).
Therefore, $\overline{AB}^2 : \overline{AC}^2 :: M : N$, (Th. 6, B. II).

PROBLEM XXIV.

To cut a line into extreme and mean ratio; that is, so that the whole line shall be to the greater part, as that greater part is to the less.

REMARK.—The geometrical solution of this problem is not immediately apparent, but it is at once suggested by the form of the equation, which a simple algebraic analysis of its conditions leads to.

Represent the line to be divided by $2a$, the greater part by x, and consequently the other, or less part, by $2a - x$.

Now, the given line and its two parts are required, to satisfy the following proportion:

$$2a : x :: x : 2a - x$$

whence, $$x^2 = 4a^2 - 2ax$$

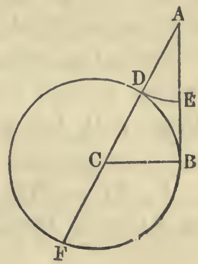
By transposition, $$x^2 + 2ax = 4a^2 = (2a)^2$$

If we add a^2 to both members of this equation, we shall have,

$$x^2 + 2ax + a^2 = (2a)^2 + a^2$$

or, $$(x + a)^2 = (2a)^2 + a^2$$

This last equation indicates that the lines represented by $(x + a)$, $2a$, and a, are the three sides of a right-angled triangle, of which $(x + a)$ is the hypotenuse, the given line, $2a$, one of the sides, and its half, a, the other.



Therefore, let AB represent the given line, and from the extremity, B, draw BC at right angles to AB, and make it equal to one half of AB.

With C, as a center, and radius CB, describe a circle. Draw AC and produce it to F. With A as a center and AD as a radius, describe the arc DE; this arc will divide the line AB, as required.

We are now to prove that

$$AB : AE :: AE : EB$$

By Th. 18, B. III, we have,

$$AF \times AD = \overline{AB}^2$$

or,

$$AF : AB :: AB : AD$$

Then, (by Cor., Th. 8, Book II), we may have,

$$(AF - AB) : AB :: (AB - AD) : AD$$

Since $CB = \frac{1}{2}AB = \frac{1}{2}DF$; therefore, $AB = LF$

Hence, $AF - AB = AF - DF = AD = AE.$

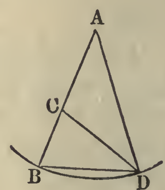
Therefore, $AE : AB :: EB : AE$

By taking the extremes for the means, we have,

$$AB : AE :: AE : EB.$$

PROBLEM XXV.

To describe an isosceles triangle, having its two equal angles each double the third angle, and the equal sides of any given length.



Let AB be one of the equal sides of the required triangle; and from the point A, with the radius AB, describe an arc, BD.

Divide the line AB into extreme and mean ratio by the last problem, and suppose C the point of division, and AC the greater segment.

From the point B, with AC, the greater segment, as a radius, describe another arc, cutting the arc BD in D. Draw BD, DC, and DA. The triangle ABD is the triangle required.

As $AC = BD$, by construction; and as AB is to AC as AC is to BC, by the division of AB; therefore

$$AB : BD :: BD : BC$$

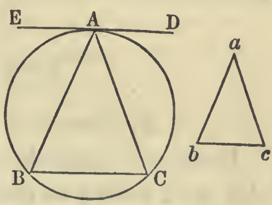
Now, as the terms of this proportion are the sides of the two triangles about the common angle, B, it follows, (Cor. 2, Th. 17, B. II), that the two triangles, ABD and

BDC, are equiangular; but the triangle ABD is isosceles; therefore, BDC is isosceles also, and $BD = DC$; but $BD = AC$: hence, $DC = AC$, (Ax. 1), and the triangle ACD is isosceles, and the $\angle CDA = \angle A$. But the exterior angle, $BCD = CDA + A$, (Th. 12, B. I). Therefore, $\angle BCD$, or its equal $\angle B = \angle CDA + \angle A$; or the angle $B = 2\angle A$. Hence, the triangle ABD has each of its angles, at the base, double of the third angle.

SCHOLIUM.—As the two angles, at the base of the triangle ABD, are equal, and each is double the angle A, it follows that the sum of the three angles is *five times* the angle A. But, as the three angles of every triangle are always equal to two right angles, or 180°, the angle A must be one fifth of two right angles, or 36°; therefore, BD is a chord of 36°, when AB is a radius to the circle; and ten such chords would extend exactly round the circle, or would form a decagon.

PROBLEM XXVI.

Within a given circle to inscribe a triangle, equiangular to a given triangle.



Let ABC be the circle, and abc the given triangle. From any point, as A, draw ED tangent to the given circle at A, (Problem 12).

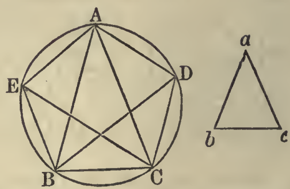
From the point A, in the line AD, lay off the angle $DAC =$ the angle b, (Problem 5), and the angle $EAB =$ the angle c, and draw BC.

The triangle ABC is inscribed in the circle; it is equiangular to the triangle abc, and hence it is the triangle required.

Proof, (Th. 12, B. III).

PROBLEM XXVII.

To inscribe a regular pentagon in a given circle.



1st. Describe an isosceles triangle, *abc*, having each of the equal angles, *b* and *c*, double the third angle, *a*, by Problem 25.

2d. Inscribe the triangle, *ABC*, in the given circle, equiangular to the triangle *abc*, by Problem 26; then each of the angles, *B* and *C*, is double the angle *A*.

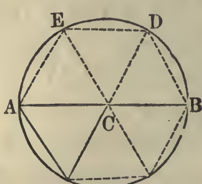
3d. Bisect the angles *B* and *C*, by the lines *BD* and *CE*, (Problem 2), and draw *AE*, *EB*, *CD*, *DA*; and the figure *AEBCD* is the pentagon required.

By construction, the angles *BAC*, *ABD*, *DBC*, *BCE*, *ECA*, are all equal; therefore, (B. III, Th. 9, Cor.), the arcs, *BC*, *AD*, *DC*, *AE*, and *EB*, are all equal; and if the arcs are equal, the chords *AE*, *EB*, etc., are equal.

SCHOLIUM.—The arc subtended by one of the sides of a regular pentagon, being one fifth of the whole circumference, is equal to $\frac{360°}{5}=72°$.

PROBLEM XXVIII.

To inscribe a regular hexagon in a circle.



Draw any diameter of the circle, as *AB*, and from one extremity, *B*, draw *BD* equal to *BC*, the radius of the circle. The arc, *BD*, will be one sixth part of the whole circumference, and the chord *BD* will be a side of the regular polygon of six sides.

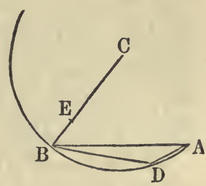
In the △ *CBD*, as $CB = CD$, and $BD = CB$ by construction, the △ is equilateral, and of course equiangular.

Since the sum of the three angles of every △ is equal to two right angles, or to 180 degrees, when the three angles are equal to one another, each one of them must be 60 degrees; but 60 degrees is a sixth part of 360 degrees, the whole number of degrees in a circle; therefore, the arc whose chord is equal to the radius, is a sixth part of the circumference; and, if a polygon of six equal sides be inscribed in a circle, each side will be equal to the radius.

SCHOLIUM. — Hence, as BD is the chord of 60°, and equal to BC or CD, we say generally, *that the chord of* 60° *is equal to radius.*

PROBLEM XXIX.

To find the side of a regular polygon of fifteen sides, which may be inscribed in any given circle.

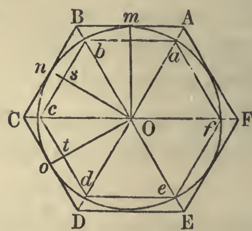


Let CB be the radius of the given circle; divide it into extreme and mean ratio, (Problem 24), and make BD equal to CE, the greater part; then BD will be a side of a regular polygon of ten sides, (Scholium to Problem 25). Draw BA = to CB, and it will be a side of a polygon of six sides. Draw DA, and that line must be the side of a polygon which corresponds to the arc of the circle expressed by $\frac{1}{6}$ less $\frac{1}{10}$, of the whole circumference; or $\frac{1}{6} - \frac{1}{10} = \frac{4}{60} = \frac{1}{15}$; that is, one-fifteenth of the whole circumference; or, DA is a side of a regular polygon of 15 sides. But the 15th part of 360° is 24°; hence the side of a regular inscribed polygon of fifteen sides is the chord of an arc of 24°.

PROBLEM XXX.

In a given circle to inscribe a regular polygon of any number of sides, and then to circumscribe the circle by a similar polygon.

Let the circumference of the circle, whose center is C, be divided into any number of equal arcs, as *amb*, *bnc*, *cod*, etc.; then will the polygon *abcde*, etc., bounded by the chords of these arcs, be regular and inscribed; and the polygon *ABCDE*, etc., bounded by the tangents to these arcs at their middle points *m*, *n*, *o*, etc, be a similar circumscribed polygon.



First. — The polygon *abcde*, etc., is equilateral, because its sides are the chords of equal arcs of the same circle, (Th. 5, B. III); and it is equiangular, because its angles are inscribed in equal segments of the same circle, (Th. 8, B. III). Therefore the polygon is regular, (Def. 14, B. III), and it is inscribed, since the vertices of all its angles are in the circumference of the circle, (Def. 13, B. III).

Second.—If we draw the radius to the point of tangency of the side *AB* of the circumscribed polygon, this radius is perpendicular to *AB*, (Th. 4, B. III), and also to the chord *ab*, (B. III, Th. 1, Cor.); hence *AB* is parallel to *ab*, and for the same reason *BC* is parallel to *bc*; therefore the angle *ABC* is equal to the angle *abc*, (Th. 8, B. I). In like manner we may prove the other angles of the circumscribed polygon, each equal to the corresponding angle of the inscribed polygon. These polygons are therefore mutually equiangular.

Again, if we draw the radii *Om* and *On*, and the line *OB*, the two △'s thus formed are right-angled, the one at *m* and the other at *n*, the side *OB* is common and *Om* is equal to *On*; hence the difference of the squares described on *OB* and *Om* is equivalent to the difference of the squares described on *OB* and *On*. But the first difference is equivalent to the square described on *Bm*, and the second difference is equivalent to the square described on Bn; hence Bm is equal to Bn, and the two right-angled triangles are equal, (Th. 21, B. I), the angle BOm opposite the side Bm being equal to the angle BOn, opposite the equal side Bn. The line OB therefore passes through the middle point of the arc mbn; but because m and n are the middle points of the equal arcs amb and bnc, the vertex of the angle abc is also at the middle point of the arc mbn. Hence the line OB, drawn from the center of the circle to the vertex of the angle ABC, also passes through the vertex of the angle abc. By precisely the same process of reasoning, we may prove that OC passes through the point c, OD through the point d, etc.; hence the lines joining the center with the vertices of the angles of the circumscribed polygon, pass through the vertices of the corresponding angles of the inscribed polygon; and conversely, the radii drawn to the vertices of the angles of the inscribed polygon, when produced, pass through the vertices of the corresponding angles of the circumscribed polygon.

Now, since ab is parallel to AB, the similar $\triangle$'s abO and ABO, give the proportion

$$Ob : OB :: ab : AB,$$

and the $\triangle$'s, bcO and BCO, give the proportion

$$Ob : OB :: bc : BC.$$

As these two proportions have an antecedent and consequent, the same in both, we have, (Th. 6, B. II),

$$ab : AB :: bc : BC.$$

In like manner we may prove that

$$bc : BC :: cd : CD, \text{ etc., etc.}$$

The two polygons are therefore not only equiangular, but the sides about the equal angles, taken in the same order, are proportional; they are therefore similar, (Def. 16. B. II).

Cor. 1. To inscribe any regular polygon in a circle, we have only to divide the circumference into as many equal parts as the polygon is to have sides, and to draw the chords of the arcs; hence, in a given circle, it is possible to inscribe regular polygons of any number of sides whatever. Having constructed any such polygon in a given circle, it is evident, that by changing the radius of the circle without changing the number of sides of the polygon, it may be made to represent any regular polygon of the same name, and it will still be inscribed in a circle. As this reasoning is applicable to regular polygons of whatever number of sides, it follows, that *any regular polygon may be circumscribed by the circumference of a circle.*

Cor. 2. Since ab, bc, cd, etc., are equal chords of the same circle, they are at the same distance from the center, (Th. 3, B. III); hence, if with O as a center, and Ot, the distance of one of these chords from that point, as a radius, a circumference be described, it will touch all of these chords at their middle points. It follows, therefore, that *a circle may be inscribed within any regular polygon.*

Scholium.—The center, O, of the circle, may be taken as the *center* of both the inscribed and circumscribed polygons; and the angle AOB, included between lines drawn from the center to the extremities of one of the sides AB, is called *the angle at the center.* The perpendicular drawn from the center to one of the sides is called the *Apothem* of the polygon.

Cor. 3. The angle at the center of any regular polygon is equal to four right angles divided by the number of sides of the polygon. Thus, if n be the number of sides of the polygon, the angle at the center will be expressed by $\frac{360°}{n}$.

Cor. 4. If the arcs subtended by the sides of any regular inscribed polygon be bisected, and the chords of these semi-arcs be drawn, we shall have a regular inscribed polygon of double the number of sides. Thus, from the square we may pass successively to regular inscribed polygons of 8, 16, 32, etc., sides. To get the corresponding circumscribed polygons, we have merely to draw tangents at the middle points of the arcs subtended by the sides of the inscribed polygons.

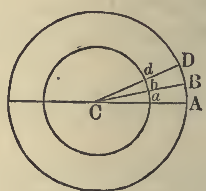
Cor. 5. It is plain that each inscribed polygon is but a part of one having twice the number of sides, while each circumscribed polygon is but a part of one having one half the number of sides

BOOK V.

ON THE PROPORTIONALITIES AND MEASUREMENT OF POLYGONS AND CIRCLES.

PROPOSITION I.—THEOREM.

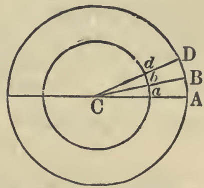
The area of any circle is equal to the product of its radius by one half of its circumference.



Let CA be the radius of a circle, and AB a very small portion of its circumference; then ACB will be a sector. We may conceive the whole circle made up of a great number of such sectors; and when each sector is very small, the arcs AB, BD, etc., each one taken separately, may be regarded as right lines; and the sectors CAB, CBD, etc., will be triangles. The triangle, ACB, is measured by the product of the base, AC, multiplied into one half the altitude, AB, (Th. 33, Book I); and the triangle BCD is measured by the product of BC, or its equal, AC, into one half BD; then the area, or measure of the two triangles, or sectors, is the product of AC, multiplied by one half of AB plus one half of BD, and so on for all the sectors that compose the circle; therefore, the area of the circle is measured *by the product of the radius into one half the circumference.*

PROPOSITION II.—THEOREM.

Circumferences of circles are to one another as their radii, and their areas are to one another as the squares of their radii.



Let CA be the radius of a circle, and Ca the radius of another circle. Conceive the two circles to be so placed upon each other so as to have a common center.

Let AB be such a certain definite portion of the circumference of the larger circle, that m times AB will represent that circumference.

But whatever part AB is of the greater circumference, the same part ab is of the smaller; for the two circles have the same number of degrees, and are of course susceptible of division into the same number of sectors. But by proportional triangles we have,

$$CA : Ca :: AB : ab$$

Multiply the last couplet by m, (Th. 4, B. II), and we have

$$CA : Ca :: m.AB : m.ab.$$

That is, *the radius of one circle is to the radius of another, as the circumference of the one is to the circumference of the other.*

To prove the second part of the theorem, let C represent the area of the larger circle, and c that of the smaller; now, whatever part the sector CAB is of the circle C, the sector Cab is the corresponding part of the circle c.

That is, $\quad C : c \quad :: CAB : Cab,$
but, $\quad CAB : Cab :: (CA)^2 : (Ca)^2,$ (Th. 20, B. II).
Therefore, $\quad C : c \quad :: (CA)^2 : (Ca)^2,$ (Th. 6, B. II).

That is, *the area of one circle is to the area of another, as* *the square of the radius of the one is to the square of the radius of the other.*

Hence the theorem.

Cor. If $C : c :: (CA)^2 : (Ca)^2$,
then, $C : c :: 4(CA)^2 : 4(Ca)^2$.

But $4(CA)^2$ is the square of the diameter of the larger circle, and $4(Ca)^2$ is the square of the diameter of the smaller. Denoting these diameters respectively by D and d, we have,

$$C : c :: D^2 : d^2.$$

That is, *the areas of any two circles are to each other, as the squares of their diameters.*

SCHOLIUM. — As the circumference of every circle, great or small, is assumed to be the measure of 360 degrees, if we conceive the circumference to be divided into 360 equal parts, and one such part represented by AB on one circle, or ab on the other, AB and ab will be very near straight lines, and the length of such a line as AB will be greater or less, according to the radius of the circle; but its *absolute* length *cannot* be determined until we know the *absolute relation* between the diameter of a circle and its circumference.

PROPOSITION III.—THEOREM.

When the radius of a circle is unity, its area and semi-circumference are numerically equal.

Let R represent the radius of any circle, and the Greek letter, π, the half circumference of a circle whose radius is unity. Since circumferences are to each other as their radii, when the radius is R, the semi-circumference will be expressed by πR.

Let m denote the area of the circle of which R is the radius; then, by Theorem 1, we shall have, for the area of this circle, $\pi R^2 = m$, which, when $R = 1$, reduces to $\pi = m$.

This equation is to be interpreted as meaning that the semi-circumference contains its unit, the radius, as many times as the area of the circle contains its unit, the square of the radius.

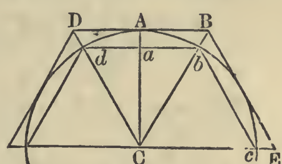
REMARK.—The celebrated problem of squaring the circle has for its object to find a line, the square on which will be equivalent to the area of a circle of a given diameter; or, in other words, it proposes to find the ratio between the area of a circle and the square of its radius.

An approximate solution only of this problem has been as yet discovered, but the approximation is so close that the exact solution is no longer a question of any practical importance.

PROPOSITION IV.—PROBLEM.

Given, the radius of a circle unity, to find the areas of regular inscribed and circumscribed hexagons.

Conceive a circle described with the radius CA, and in this circle inscribe a regular polygon of six sides (Prob. 28, B. IV), and each side will be equal to the radius CA; hence, the whole *perimeter* of this polygon must be six times the radius of the circle, or three times the diameter. The chord bd is bisected by CA. Produce Cb and Cd, and through the point A, draw BD parallel to bd; BD will then be a side of a regular polygon of six sides, circumscribed about the circle, and we can compute the length of this line, BD, as follows: The two triangles, Cbd and CBD, are equiangular, by construction; therefore,



$$Ca : bd :: CA : BD.$$

Now, let us assume $CA = Cd =$ the radius of the circle, equal unity; then $bd = 1$, and the preceding proportion becomes

$$Ca : 1 :: 1 : BD \qquad (1)$$

In the right-angled triangle Cad, we have,

$$(Ca)^2 + (ad)^2 = (Cd)^2, \qquad \text{(Th. 39, B. I).}$$

That is, $(Ca)^2 + \frac{1}{4} = 1$, because $Cd = 1$, and $ad = \frac{1}{2}$.

Whence, $Ca = \frac{1}{2}\sqrt{3}$. This value of Ca, substituted in proportion (1), gives

$$\frac{1}{2}\sqrt{3} : 1 :: 1 : BD; \text{ hence, } BD = \frac{2}{\sqrt{3}}.$$

But the area of the triangle Cbd is equal to $bd\,(=1,)$ multiplied by $\frac{1}{2}Ca = \frac{1}{4}\sqrt{3}$; and the area of the triangle CBD is equal to BD multiplied by $\frac{1}{2}CA$.

Whence, area, $Cbd = \frac{1}{4}\sqrt{3}$,

and, area, $CBD = \frac{1}{\sqrt{3}}$.

But the area of the inscribed polygon is six times that of the triangle Cbd, and the area of the circumscribed polygon is six times that of the triangle CBD.

Let the area of the inscribed polygon be represented by p, and that of the circumscribed polygon by P.

$$\text{Then } p = \frac{3}{2}\sqrt{3}, \text{ and } P = \frac{6}{\sqrt{3}} = \frac{2 \times 3}{\sqrt{3}} = 2\sqrt{3}.$$

$$\text{Whence } p : P :: \frac{3}{2}\sqrt{3} : 2\sqrt{3} :: \frac{3}{2} : 2 :: 3 : 4 :: 9 : 12$$

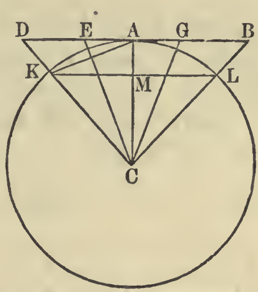
$$p = \frac{3}{2}\sqrt{3} = 2.59807621. \quad P = 2\sqrt{3} = 3.46410161.$$

Now, it is obvious that the *area* of the circle must be included between the areas of these two polygons, and not far from, but somewhat greater than, their half sum, which is 3.03 +; and this may be regarded as the first approximate value of the area of the circle to the radius unity.

PROPOSITION V.—PROBLEM.

Given, the areas of two regular polygons of the same number of sides, the one inscribed in and the other circumscribed about, the same circle, to find the areas of regular inscribed and circumscribed polygons of double the number of sides.

Let p represent the area of the given inscribed polygon, and P that of the circumscribed polygon of the same number of sides. Also denote by p' the area of the inscribed polygon of double the number of sides, and by P' that of the corresponding circumscribed polygon. Now, if the arc KAL be some exact part, as one-fourth, one fifth, etc., of the circumference of the circle, of which C is the center and CA the radius, then will KL be the side of a regular inscribed polygon, and the triangle KCL will be the same part of the whole polygon that the arc KAL is of the whole circumference, and the triangle CDB will be a like part of the circumscribed polygon. Draw CA to the point of tangency, and bisect the angles ACB and ACD, by the lines CG and CE, and draw KA.



It is plain that the triangle ACK is an exact part of the inscribed polygon of double the number of sides, and that the $\triangle$ ECG is a like part of the circumscribed polygon of double the number of sides. Represent the area of the $\triangle$ LCK by a, and the area of the $\triangle$ BCD by b, that of the $\triangle$ ACK by x, and that of the $\triangle$ ECG by y, and suppose the $\triangle$'s, KCL and DBC, to be each the nth part of their respective polygons.

Then, $\quad na = p, \quad nb = P, \quad 2nx = p',$

and, $\quad 2ny = P';$

But, by (Th. 33, B. I), we have

$$CM \,.\, MK = a \qquad (1)$$
$$CA \,.\, AD = b \qquad (2)$$
$$CA \,.\, MK = 2x \qquad (3)$$

Multiplying equations (1) and (2), member by member, we have

$$(CM \,.\, AD) \times (CA \,.\, MK) = ab \qquad (4)$$

From the similar $\triangle$'s CMK and CAD, we have

$$CM : MK :: CA : AD$$

whence $$CM \,.\, AD = CA \,.\, MK$$

But from equation (3) we see that each member of this last equation is equal to $2x$; hence equation (4) becomes

$$2x \,.\, 2x = ab$$

If we multiply both members of this by $n^2 = n \,.\, n$, we shall have

$$4n^2x^2 = na.nb = p.P$$

or, taking the square root of both members,

$$2nx = \sqrt{p.P}$$

That is, *the area of the inscribed polygon of double the number of sides is a mean proportional between the areas of the given inscribed and circumscribed polygons* p *and* P.

Again, since CE bisects the angle ACD, we have, by, (Th. 24, B. II),

$$\begin{aligned} AE : ED &:: CA : CD \\ &:: CM : CK \\ &:: CM : CA \end{aligned}$$

hence, $$AE : AE + ED :: CM : CM + CA.$$

Multiplying the first couplet of this proportion by CA, and the second by MK, observing that $AE + ED = AD$, we shall have

$$AE.CA : AD.CA :: CM.MK : (CM + CA)\,MK.$$

But $AE.CA$ measures the area of the $\triangle\ CEG$, which we have called y, $AD.CA = \triangle\ CBD = b$, $CM.MK = \triangle\, CKL = a$, and $(CM + CA)MK = \triangle\, CKL + 2\triangle\, CAK = a + 2x$, as is seen from equations (1) and (3). Therefore, the above proportion becomes

$$y : b :: a : a + 2x.$$

Multiplying the first couplet by $2n$, and the second by n, we shall have

$$2ny : 2nb :: na : na + 2nx$$

That is, $$P' : 2P :: p : p + p'$$

whence, $$P' = \frac{2Pp}{p + p'}$$

and as the value of p' has been previously found equal to $\sqrt{Pp}$, the value of P' is known from this last equation, and the problem is completely solved.

PROPOSITION VI.—PROBLEM.

To determine the approximate numerical value of the area of a circle, when the radius is unity.

We have now found, (Prob. 4), the areas of regular inscribed and circumscribed hexagons, when the radius of the circle is taken as the unit; and Prob. 5 gives us formulæ for computing from these the areas of regular inscribed and circumscribed polygons of twelve sides, and from these last we may pass to polygons of twenty-four sides, and so on, without limit. Now, it is evident that, as the number of sides of the inscribed polygon is increased, the polygon itself will increase, gradually approaching the circle, which it can never surpass. And it is equally evident that, as the number of sides of the circumscribed polygon is increased, the polygon itself will decrease, gradually approaching the circle, less than which it can never become.

The circle being included between any two corresponding inscribed and circumscribed polygons, it will differ from either less than they differ from each other; and the area of either polygon may then be taken as the area of the circle, from which it will differ by an amount less than the difference between the polygons.

It is also plain that, as the areas of the polygons approach equality, their perimeters will approach coincidence with each other, and with the circumference of the circle.

Assuming the areas already found for the inscribed and circumscribed hexagons, and applying the formulæ of Prob. 5 to them and to the successive results obtained, we may construct the following table:

NUMBER OF SIDES.	INSCRIBED POLYGONS.	CIRCUMSCRIBED POLYGONS.
6	$\frac{3}{2}\sqrt{3} = 2.59807621$	$2\sqrt{3} = 3.46410161$
12	$3 = 3.0000000$	$\frac{12}{2+\sqrt{3}} = 3.2153904$
24	$\frac{6}{\sqrt{2+\sqrt{3}}} = 3.1058286$	3.1596602
48	3.1326287	3.1460863
96	3.1393554	3.1427106
192	3.1410328	3.1418712
384	3.1414519	3.1416616
768	3.1415568	3.1416092
1536	3.1415829	3.1415963
3072	3.1415895	3.1415929
6144	3.1415912	3.1415927

Thus we have found, that when the radius of a circle is 1, the semi-circumference must be more than 3.1415912, and less than 3.1415927; and this is as accurate as can be determined with the small number of decimals here used. To be more accurate we must have more decimal places, and go through a very tedious mechanical operation; but this is not necessary, for the result is well known, and is 3.1415926535897, *plus* other decimal places to the 100th, without termination. This result was discovered through the aid of an infinite series in the Differential and Integral Calculus.

The number, 3.1416, is the one generally used in practice, as it is much more convenient than a greater number of decimals, and it is sufficiently accurate for all ordinary purposes.

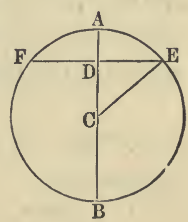
In analytical expressions it has become a general custom with mathematicians to represent this number by the Greek letter π, and, therefore, when any diameter of a circle is represented by D, the circumference of the same circle must be πD. If the radius of a circle is represented by R, the circumference must be represented by $2\pi R$.

SCHOLIUM. — The side of a regular inscribed hexagon subtends an arc of 60°, and the side of a regular polygon of twelve sides subtends an arc of 30°; and so on, the length of the arc subtended by the sides of the polygons, varying inversely with the number of sides.

Angles are measured by the arcs of circles included between their sides; they may also be measured by the chords of these arcs, or rather by the half chords called *sines* in Trigonometry. For this purpose, it becomes necessary to know the length of the chord of every possible arc of a circle.

PROPOSITION VII.—PROBLEM.

Given, the chord of any arc, to find the chord of one half that arc, the radius of the circle being unity.



Let FE be the given chord, and draw the radii CA and CE, the first perpendicular to FE, and the second to its extremity, E.

Denote FE by $2c$, and the chord of the half arc AE by x.

Then, in the right-angled triangle, DCE, we have $\overline{DC}^2 = \overline{CE}^2 - \overline{DE}^2$. Whence, since $CE = 1$, $DC = \sqrt{1 - c^2}$.

If from $CA = 1$ we subtract DC, we shall have AD. That is, $AD = 1 - \sqrt{1 - c^2}$; but $\overline{AD}^2 + \overline{DE}^2 = \overline{AE}^2$, and $\overline{AD}^2 = 2 - 2\sqrt{1 - c^2} - c^2$. Adding to the first member of this last equation $\overline{DE}^2$, and to the second its value c^2, we have

$$\overline{AD}^2 + \overline{DE}^2 = 2 - 2\sqrt{1 - c^2}.$$

Whence, $\quad AE = \sqrt{2 - 2\sqrt{1 - c^2}}$, the value sought.

By applying this formula successively to any known chord, we can find the chord of one half the arc, that of half of the half, and so on, to the chords of the most minute arcs.

Application.

The greatest chord in a circle is its diameter, which is 2 when the radius is 1; therefore, we may commence by making $2c = 2$, and $c = 1$.

Then, $AE = \sqrt{2 - 2\sqrt{1 - c^2}} = \sqrt{2 - 2\sqrt{1-1}} = \sqrt{2} =$ 1.41421356, which is the chord of 90°.

Now make $2c = 1.41421356$, and $c = .70710678 = \frac{1}{2}\sqrt{2}$.

We shall then have,

chord of $45° = \sqrt{2 - \sqrt{2}} = \sqrt{2 - 1.41421356} = \sqrt{.58578644} =$ $.7653+$.

Again, placing $2c = .7653+$, and applying the formula, we can obtain the chord of 22° 30′, and from this the chord of 11° 15′, and so on, as far as we please.

We may take, for another starting point, the chord of 60°, which is known to be equal to the radius of the circle,(Prob. 26, B. IV). If, as above, we make successive applications of the formula, putting first $2c = 1$, we shall arrive at the results in the following

TABLE.

Chord of 60°,		$= \frac{1}{6}$	of a circumference,			1.0000000000
"	" 30°,	$= \frac{1}{12}$	"	"		.5176380902
"	" 15°,	$= \frac{1}{24}$	"	"		.2610523842
"	" 7° 30′,	$= \frac{1}{48}$	"	"		.1308062583
"	" 3° 45′,	$= \frac{1}{96}$	"	"		.0654381655
"	" 1° 52′ 30″,	$= \frac{1}{192}$	"	"		.0327234632
"	" 56′ 15″,	$= \frac{1}{384}$	"	"		.0163622792
"	" 28′ 7″ 30‴,	$= \frac{1}{768}$	"	"		.0081812080
"	" 14′ 3″ 45‴,	$= \frac{1}{1536}$	"	"		.0040906112
"	" 7′ 1″ 52½‴,	$= \frac{1}{3072}$	"	"		.0020453068
	etc.	etc.				

It is obvious that an arc so small as seven minutes of a degree can differ but very little from its chord; therefore, if we take .002045307 to be the true value of the $\frac{1}{3072}$ of the circumference, the whole circumference must be the product of .002045307 by 3072, which is 6.283183104 = circumference whose radius is unity. The half of this, 3.141592552, is the semi-circumference, the more exact value of which, as stated, (Prop. 6), is 3.141592653.

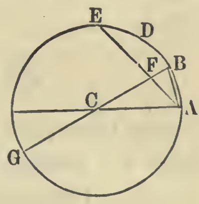
The value of the half circumference being now determined, if that of any arc whatever be required, we have merely to divide 3.141592, etc., by 10800, the number of minutes in a semi-circumference, and multiply the quotient by the number of minutes in the arc whose length is required.

But this investigation has been carried far enough for our present purposes. It will be resumed under the subject of Trigonometry.

We insert the following beautiful theorem for the trisection of an arc, although not necessary for practical application. Those not acquainted with cubic equations may omit it.

PROPOSITION VIII.—THEOREM.

Given, the chord of any arc, to determine the chord of one third of such arc.



Let AE be the given chord, and conceive its arc divided into three equal parts, as represented by AB, BD, and DE.

Through the center draw BCG, and draw AB. The two $\triangle$'s, CAB and ABF, are equiangular; for, the angle FAB, being at the circumference, is measured by one half the arc BE, which is equal to AB, and the angle BCA, being at the center, is measured by the arc AB; therefore, the angle FAB = the angle BCA; but the angle CBA or FBA, is common to both triangles; therefore, the third angle, CAB, of the one triangle, is equal to the third angle, AFB, of the other, (Th. 12, B. I, Cor. 2), and the two triangles are equiangular and similar.

But the $\triangle\ ACB$ is isosceles; therefore, the $\triangle\ AFB$ is also isosceles, and $AB = AF$, and we have the following proportions:

$$CA : AB :: AB : BF.$$

Now, let $AE = c$, $AB = x$, $AC = 1$. Then $AF = x$, and $EF = c - x$, and the proportion becomes,

$$1 : x :: x : BF. \quad \text{Hence, } BF = x^2.$$

Also, $$FG = 2 - x^2.$$

As AE and BG are two chords intersecting each other at the point F, we have,

$$GF \times FB = AF \times FE, \text{ (Th. 17, B. III).}$$

That is, $(2 - x^2)\ x^2 = x\ (c - x)$;

or, $$x^3 - 3x = -c.$$

If we suppose the arc AE to be 60 degrees, then $c = 1$, and the equation becomes $x^3 - 3x = -1$; a cubic equation, easily resolved by Horner's method, (Robinson's New University Algebra, Art. 464), giving $x = .347296 +$ the chord of 20°. This again may be taken for the value of c, and a second solution will give the chord of 6° 40′, and so on, trisecting successively as many times as we please.

PRACTICAL PROBLEMS.

The theorems and problems with which we have been thus far occupied, relate to plane figures; that is, to figures all of whose parts are situated in the same plane. It yet remains for us to investigate the intersections and relative positions of planes; the relations and positions of lines with reference to planes in which they are not contained; and the measurements, relations, and properties of solids, or volumes. But before we proceed to this, it is deemed advisable to give some practical problems for the purpose of exercising the powers of the student, and of fixing in his mind those general geometrical principles with which we must now suppose him to be acquainted.

1. The base of an isosceles triangle is 6, and the opposite angle is 60°; required the length of each of the other two equal sides, and the number of degrees in each of the other angles.

2. One angle of a right-angled triangle is 30°; what is the other angle? Also, the least side is 12, what is the hypotenuse?

Ans. { The hypotenuse is 24, the double of the least side. Why?

3. The perpendicular distance between two parallel lines is 10; what angles must a line of 20 make with these parallels to extend exactly from the one to the other? *Ans.* The angles must be 30° and 150°.

4. The perpendicular distance between two parallels is 20 feet, and a line is drawn across them at an angle of 45°; what is its length between the parallels?

Ans. $20\sqrt{2}$.

5. Two parallels are 8 feet asunder, and from a point in one of the parallels two lines are drawn to meet the other; the length of one of these lines is 10 feet, and that of the other 15 feet; what is the distance between the points at which they meet the other parallel?

Ans. 6.69 ft., or 18.69 ft. (See Th. 39, B. I).

6. Two parallels are 12 feet asunder, and, from a point on one of them, two lines, the one 20 feet and the other 18 feet in length, are drawn to the other parallel; what is the distance between the two lines on the other parallel, and what is the area of the triangle so formed?

Ans. { The distance on the other parallel is 29.416 feet, or 2.584 feet; and the area of the triangle is 176.496, or 15.504 square feet.

7. The diameter of a circle is 12, and a chord of the

circle is 4; what is the length of the perpendicular drawn from the center to this chord? (See Th. 3, B. III).

Ans. $4\sqrt{2}$.

8. Two parallel chords in a circle were measured and found to be 8 feet each, and their distance asunder was 6 feet; what was the radius of the circle?

Ans. 5 feet.

9. Two chords on opposite sides of the center of a circle are parallel, and one of them has a length of 16 and the other of 12 feet, the distance between them being 14 feet. What is the diameter of the circle?

Ans. 20 feet.

10. An isosceles triangle has its two equal sides, 15 each, and its base 10. What must be the altitude of a right-angled triangle on the same base, and having an equal area?

11. From the extremities of the base of any triangle, draw lines bisecting the other sides; these two lines intersecting within the triangle, will form another triangle on the same base. How will the area of this new triangle compare with that of the whole triangle?

Ans. Their areas will be as 3 to 1.

12. Two parallel chords on the same side of the center of a circle, whose diameter is 32, are measured and found to be, the one 20, and the other 8. How far are they asunder? *Ans.* $\sqrt{240} - \sqrt{156} = 3 +$.

If we suppose the two chords to be on opposite sides of the center, their distance apart will then be $\sqrt{240} + \sqrt{156} = 15.49 + 12.49 = 27.98$.

13. The longer of the two parallel sides of a trapezoid is 12, the shorter 8, and their distance asunder 5. What is the area of the trapezoid? and if we produce the two inclined sides until they meet, what will be the area of the triangle so formed?

Ans. Area of trapezoid, 50; area of triangle, 40; area of triangle and trapezoid, 90.

14. The base of a triangle is 697, one of the sides is 534, and the other 813. If a line be drawn bisecting the angle opposite the base, into what two parts will the bisecting line divide the base? (See Th. 24, B. II).

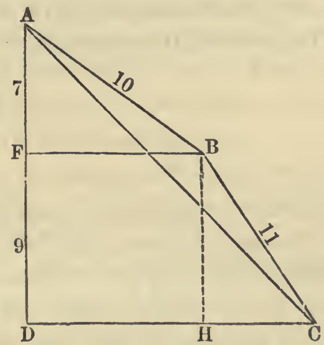
Ans. $\begin{cases} \text{The greater part will be } 420.684; \\ \text{The less} \quad \text{"} \quad \text{"} \quad 276.316. \end{cases}$

15. Draw three horizontal parallels, making the distance between the two upper parallels 7, and that between the middle and lower parallels 9; then place between the upper parallels a line equal to 10, and from the point in which it meets the middle parallel draw to the lower a line equal to 11, and join the point in which this last line meets the lower parallel, with the point in the upper parallel, from which the line 10 was drawn. Required the length of this line, and the area of the triangle formed by it and the two lines 10 and 11.

The adjoining figure will illustrate. Let A be the point on the upper parallel from which the line 10 is drawn. Then, $AF = 7$, $AB = 10$, $FB = \sqrt{100 - 49} = \sqrt{51}$.

$BH = FD = 9$, $BC = 11$, $HC = \sqrt{121 - 81} = \sqrt{40}$.

Whence, $DC = \sqrt{51} + \sqrt{40}$.



$$\overline{AC}^2 = (\sqrt{51} + \sqrt{40})^2 + (16)^2;\ AC = 20.89, \textit{Ans.}$$

The area of the triangle, ABC, can be determined by first finding the area of the trapezoid, $ABHD$, then the area of the triangle, BHC, and from their sum subtracting the area of the triangle, ADC.

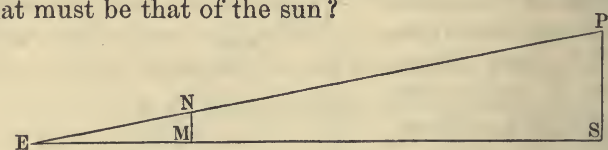
16. Construct a triangle on a base of 400, one of the angles at the base being 80°, and the other 70°; and

determine the third angle, and the area of the triangle thus constructed.

Ans. The third angle is 30°, and as nearly as our *scale of equal parts* can determine for us, the side opposite the angle 80° is 787, and that opposite 70° is 740.

The exact solution of problems like the last, except in a few particular cases, requires a knowledge of certain lines depending on the angles of the triangle. The properties and values of these lines are investigated in trigonometry; and as we are not yet supposed to be acquainted with them, we must be content with the approximate solutions obtained by the constructions and measurements made with the plane scale.

17. If we call the mean radius of the earth 1, the mean distance of the moon will be 60; and as the mean distance of the sun is 400 times the distance of the moon, its distance will be 400 times 60. The sun and moon appear to have the same diameter; supposing, then, the real diameter of the moon to be 2160 miles, what must be that of the sun?



Let E be the center of the earth, M that of the moon, and S that of the sun, and suppose ENP to be a line from the center of the earth, touching the moon and the sun.

Then, $EM : MN :: ES : SP$;

but MN is the radius of the moon, and SP that of the sun. Multiplying the consequents by 2, the above proportion becomes

$$EM : 2MN :: ES : 2SP;$$

or in numbers, $60 : 2160 :: 400 \times 60 : 2SP$;

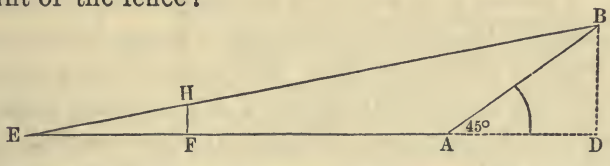
whence, $2SP =$ sun's diameter $= 864000$ miles, *Ans.*

18. In Problem 15, suppose BC to be drawn on the other side of BH, what, then, will be the value of AC, and what the area of the triangle ACB?

Ans $\begin{cases} AC = 16{,}021; \\ \text{Area of triangle, } \tfrac{1}{2}(9\sqrt{51}+7\sqrt{40}). \end{cases}$

19. A man standing 40 feet from a building which was 24 feet wide, observed that when he closed one eye, the width of the building just eclipsed or hid from view 90 rods of fence which was parallel to the width of the building; what was the distance from the eye of the observer to the fence? *Ans.* 2475 feet.

20. Taking the same data as in the last problem, except that we will now suppose the direction of the fence to be inclined at an angle of 45° to the side of the building which we see; what, in this case, must be the distance between the eye of the observer and the remoter point of the fence?



Let HF be the width of the house, E the position of the eye, and AB that of the fence. Draw BD perpendicular to EA produced; then, since the triangle ABD is right-angled and isosceles, we have $AD = DB$, and $2\overline{AD}^2 = \overline{AB}^2 = (90)^2$; $BD = 63.64$ rods, and the similar triangles EFH and EDB give the proportion

$$HF : EF :: BD : ED = 1750.1 \text{ feet};$$

and from this we find

$$\overline{EB}^2 = \overline{ED}^2 + \overline{BD}^2 = (63.64 \times \tfrac{33}{2})^2 + (1750.1)^2$$

Whence $EB = 2040.94 +$ *Ans.*

21. In a right-angled triangle, ABC, we have $AB = 493$, $AC = 1425$, and $BC = 1338$; it is required to divide this triangle into parts by a line parallel to AB, whose areas are to each other as 1 is to 3. How will the sides AC and BC be divided by this line? (See Th. 20, B. II).

Ans. Into equal parts.

22. In a right-angled triangle, ABC, right-angled at B, the base AB is 320, and the angle A is 60°; required the remaining angle and the other sides.

$$Ans. \begin{cases} \text{The angle } C = 30°; \\ AC = 640;\ BC = 554.24. \end{cases}$$

23. A hunter, wishing to determine his distance from a village in sight, took a point and from it laid off two lines in the direction of two steeples, which he supposed equally distant from him, and which he knew to be 100 rods asunder. At the distance of 50 feet on each line from the common point, he measured the distance between the lines, and found it to be 5 feet 8 inches. How far was he from the steeples?

5 ft. 8 in. : 100 rods : : 50 ft. : distance.

or, $68 : 100 \times \frac{33}{2} \times 12 : : 50 :$ distance.

Ans. $\left\{ \begin{array}{l} \text{14,559 feet,} \\ \text{or nearly} \\ \text{3 miles.} \end{array} \right.$

24. A person is in front of a building which he knows to be 160 feet long, and he finds that it covers 10 minutes of a degree; that is, he finds that the two lines drawn from his eye to the extremities of the building include an angle of 10 minutes. What is his distance from the building?

Ans. $\left\{ \begin{array}{l} \text{55,004 feet, or} \\ \text{more than 10 miles.} \end{array} \right.$

REMARK.—The questions of distance, with which we are at present occupied, depend for their solution on the properties of similar triangles. In the preceding example we apparently have but one triangle, but we have in fact two; the second being formed by the distances *unity* on the lines drawn from the eye of the observer, and the line which connects the extremities of these units of distance. This last line may be regarded as the chord of the arc 10 minutes to the radius unity. We have seen that the length of the arc 180° to the radius 1, is 3.1415926; hence the chord of 1° or 60′ is 0.017453, and of 10′ it must be 0.0029089. Therefore, by similar triangles, we have

$$0.0029089 : 160 :: 1 : Ans. = \frac{160000}{2.9089}.$$

25. In the triangle, ABC, we have given the angles $A = 32°$, and $B = 84°$. The side AB is produced, and the exterior angle CBD thus formed, is bisected by the line BE, and the angle A is also bisected by the line AE, BE and AE meeting in the point E. What is the angle C, and what is the relation between the angles C and E?

Ans. $C = 64°$; $E = \frac{1}{2}C$.

26. Suppose a line to be drawn in any direction between two parallels. Bisect the two interior angles thus formed on either side of the connecting line, and prove that the bisecting lines meet each other at right angles, and that they are the sides of a right-angled triangle of which the line connecting the parallels is the hypotenuse.

27. If the two diagonals of a trapezoid be drawn, show that two similar triangles will be formed, the parallel sides of the trapezoid being homologous sides of the triangles. What will be the relative areas of these triangles?

Ans. The triangles will be to each other as the squares on the parallel sides of the trapezoid.

28. If from the extremities of the base of any triangle, lines be drawn to any point within the triangle, forming with the base another triangle; how will the vertical angle in this last triangle compare with that in the original triangle?

Ans. It will be as much greater than the angle in the original triangle as the sum of angles at the base of the new triangle is less than the sum of those at the base of the first.

29. The two parallel sides of a trapezoid are 12 and 20, respectively, and their perpendicular distance is 8. If a line whose length is 14.5 be drawn between the inclined sides and parallel to the parallel sides, what is the area of the trapezoid, and what the area of each part, respectively, into which the trapezoid is divided?

Ans Area of the whole, 128 square units;
" smaller part, $33\frac{1}{8}$ "
" larger " $94\frac{7}{8}$ "
Dividing line at the distance of $2\frac{1}{2}$ from shorter parallel side.

30. If we assume the diameter of the earth to be

13 *

7956 miles, and the eye of an observer be 40 feet above the level of the sea, how far distant will an object be, that is just visible on the earth's surface. (Employ Th. 18, B. III, after reducing miles to feet.)

Ans. 40992 feet = 7 miles 4032 feet.

31. The diameter of a circle is 4; what is the area of the inscribed equilateral triangle? *Ans.* $3\sqrt{3}$.

32. Three brothers, whose residences are at the vertices of a triangular area, the sides of which are severally 10, 11, and 12 chains, wish to dig a well which shall be at the same distance from the residence of each. Determine the point for the well, and its distance from their residences.

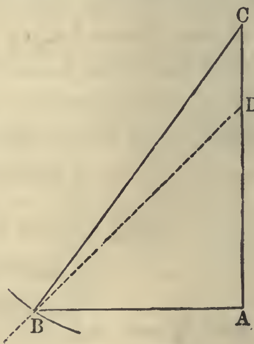
REMARK. — Construct a triangle, the sides of which are, respectively, 10, 11, and 12. The sides of this triangle will be the chords of a circle whose radius is the required distance. To find the center of this circle, bisect either two of the sides of the triangle by perpendiculars, and their intersection will be the center of the circle, and the location of the well.

Ans. The well is distant 6.405 chains, nearly, from each residence.

33. The base of an isosceles triangle is 12, and the equal sides are 20 each. What is the length of the perpendicular from the vertex to the base; and what the area of the triangle?

Ans. Perpendicular, 19.07; area, (19.07) × 6.

34. The hypotenuse of a right-angled triangle is 45 inches, and the difference between the two sides is 8.45 inches. Construct the triangle.



Suppose the triangle drawn and represented by ABC, DC being the difference between the two sides.

Now, by inspection, we discover the steps to be taken for the construction of the triangle As $AD = AB$, the angle ADB, must be equal to the angle DBA, and each equal to 45°.

Therefore, draw any line, AC, and from an assumed point in it as D, draw BD, making the angle $ADB = 45°$. Take from a scale of equal parts, 8.45 inches, and lay them off from D to C, and with C as a center, and $CB = 45$ inches as a radius, describe an arc cutting BD in B. Draw CB, and from B, draw BA at right angles to AC; then is ABC the triangle sought.

Ans. $AB = 27.3$; $AC = 35.76$, when carefully constructed.

35. Taking the same triangle as in the last problem, if we draw a line bisecting the right angle, where will it meet the hypotenuse?

Ans. 19.5 from B; and 25.5 from C.

36. The diameters of the hind and fore wheels of a carriage, are 5 and 4 feet, respectively; and their centers are 6 feet asunder. At what distance from the fore wheels will the line, passing through their centers, meet the ground, which is supposed level? *Ans.* 24 feet.

37. If the hypotenuse of a right-angled triangle is 35, and the side of its inscribed square 12, what are its sides?

Ans. 28 and 21.

38. What are the sides of a right-angled triangle having the least hypotenuse, in which if a square be inscribed, its side will be 12?

Ans. { The sides are equal to 24 each, and the least hypotenuse is double the diagonal of the square.

39. The radius of a circle is 25; what is the area of a sector of 50°?

Remark. — First find the length of an arc of 50° in a circle whose radius is unity. Then 25 times that will be the length of an arc of the same number of degrees in a circle of which the radius is 25.

Length of arc 1° radius unity $= \dfrac{3.14159265}{180}$.

" " 50° " " $= \dfrac{1.04719755}{6} \times 5$.

Area of sector $= \dfrac{1.04719755}{6} \times 125 \times \dfrac{25}{2} = 272.7077$. *Ans.*

BOOK VI.

ON THE INTERSECTIONS OF PLANES, AND THE RELATIVE POSITIONS OF PLANES AND OF PLANES AND LINES.

DEFINITIONS.

A **Plane** has been already defined to be a surface, such that the straight line which joins any two of its points will lie entirely in that surface. (Def. 9, page 9.)

1. The **Intersection** or **Common Section** of two planes is the line in which they meet.

2. A **Perpendicular to a Plane** is a line which makes right angles with every line drawn in the plane through the point in which the perpendicular meets it; and, conversely, the plane is perpendicular to the line. The point in which the perpendicular meets the plane is called the *foot* of the perpendicular.

3. A **Diedral Angle** is the separation or divergence of two planes proceeding from a common line, and is measured by the angle included between two lines drawn one in each plane, perpendicular to their common section at the same point.

The common section of the two planes is called the *edge* of the angle, and the planes are its *faces*.

4. **Two Planes are perpendicular to each other,** when their diedral angle is a right angle.

5. A **Straight Line is parallel to a plane,** when it will not meet the plane, however far produced.

6. Two Planes are parallel, when they will not intersect, however far produced in all directions.

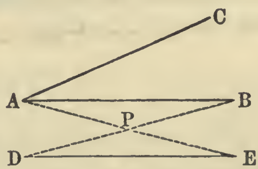
7. A Solid or Polyedral Angle is the separation or divergence of three or more plane angles, proceeding from a common point, the two sides of each of the plane angles being the edges of diedral angles formed by these plane angles.

The common point from which the plane angles proceed is called the *vertex* of the solid angle, and the intersections of its bounding planes are called its *edges.*

8. A Triedral Angle is a solid angle formed by three plane angles.

THEOREM I.

Two straight lines which intersect each other, two parallel straight lines, and three points not in the same straight line, will severally determine the position of a plane.

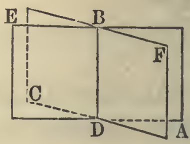


Let AB and AC be two lines intersecting each other at the point A; then will these lines determine a plane. For, conceive a plane to be passed through AB, and turned about AB as an axis until it contains the point C in the line AC. The plane, in this position, contains the lines AB and AC, and will contain them in no other. Again, let AB and DE be two parallel straight lines, and take at pleasure two points, A and B, in the one, and two points, D and E, in the other, and draw AE and BD. The last lines, AB, AE, or the lines AB, DB from what precedes, determine the position of the parallels AB, DE. And again, if A, B, and C be three points not in the same straight line, and we draw the lines AB and AC, it follows, from the first part of this proposition, that these points fix the plane.

Cor. A straight line and a point out of it determine the position of a plane.

THEOREM II.

If two planes meet each other, their common points will be found in, and form one straight line.

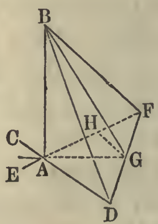


Let B and D be any two of the points common to the two planes, and join these points by the straight line BD; then will BD contain all the points common to the two planes, and be their intersection. For, suppose the planes have a common point out of the line BD; then, (Cor. Th. 1), since a straight line and a point out of it determine a plane, there would be two planes determined by this one line and single point out of it, which is absurd. Hence the common section of two planes is a straight line.

Remark.—The truth of this proposition is implicitly assumed in the definitions of this Book.

THEOREM III.

If a straight line stand at right angles to each of two other straight lines at their point of intersection, it will be at right angles to the plane of those lines.



Let AB stand at right angles to EF and CD, at their point of intersection A. Then AB will be at right angles to any other line drawn through A in the plane, passing through EF, CD, and, of course, at right angles to the plane itself. (Def. 2.)

Through A, draw any line, AG, in the plane EF, CD, and from any point G, draw GH parallel to AD. Take $HF = AH$, and join F and G and produce FG to D. Because HG is parallel to AD, we have

$$FH : HA :: FG : GD.$$

But, in this proportion, the first couplet is a ratic of equality; therefore the last couplet is also a ratio of equality,

That is, $FG = GD$, or the line FD is bisected in G.

Draw BD, BG, and BF.

Now, in the triangle AFD, as the base FD is bisected in G, we have,

$$\overline{AF}^2 + \overline{AD}^2 = 2\overline{AG}^2 + 2\overline{GF}^2 \quad (1) \quad \text{(Th. 42, B. I).}$$

Also, as DF is the base of the $\triangle$ BDF, we have by the same theorem,

$$\overline{BF}^2 + \overline{BD}^2 = 2\overline{BG}^2 + 2\overline{GF}^2 \qquad (2)$$

By subtracting (1) from (2), and observing that $\overline{BF}^2 - \overline{AF}^2 = \overline{AB}^2$, because BAF is a right angle; and $\overline{BD}^2 - \overline{AD}^2 = \overline{AB}^2$, because BAD is a right angle, we shall have,

$$\overline{AB}^2 + \overline{AB}^2 = 2\overline{BG}^2 - 2\overline{AG}^2.$$

Dividing by 2, and transposing $\overline{AG}^2$, and we have,

$$\overline{AB}^2 + \overline{AG}^2 = \overline{BG}^2.$$

This last equation shows that BAG is a right angle. But AG is any line drawn through A, in the plane EF, CD; therefore AB is at right angles to any line in the plane, and, of course, at right angles to the plane itself

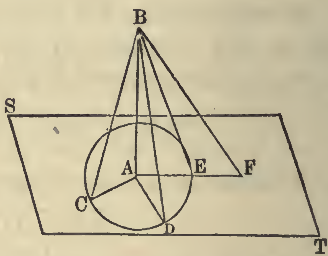
Cor. 1. The perpendicular BA is shorter than any of the oblique lines BF, BG, or BD, drawn from the point B to the plane; hence it is the shortest distance from a point to a plane.

Cor. 2. But one perpendicular can be erected to a plane from a given point in the plane; for, if there could be two, the plane of these perpendiculars would intersect the given plane in some line, as AG, and both the perpendiculars would be at right angles to this intersection at the same point, which is impossible.

Cor. 3. But one perpendicular can be let fall from a given point out of a plane on the plane; for, if there can be two, let BG and BA be such perpendiculars, then would the triangle BAG be right angled at both A and G, which is impossible.

THEOREM IV.

If from any point of a perpendicular to a plane, oblique lines be drawn to different points in the plane, those oblique lines which meet the plane at equal distances from the foot of the perpendicular are equal; and those which meet the plane at unequal distances from the foot of the perpendicular are unequal, the greater distances corresponding to the longer oblique lines.



Take any point B in the perpendicular BA to the plane ST, and draw the oblique lines BC, BD, and BE, the points C, D, and E, being equally distant from A, the foot of the perpendicular. Produce AE to F, and draw BF; then will $BC = BD = BE$, and $BF > BE$.

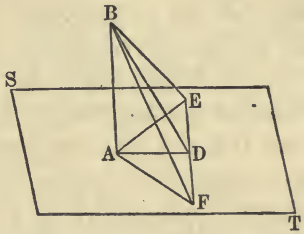
For, the triangles BAC, BAD, and BAE are all right-angled at A, the side BA is common, and $AC = AD = AE$ by construction, hence, (Th. 16, B. I), $BC = BD = BE$. Moreover, since $AF > AE$, the oblique line $BF > BE$.

Cor. If any number of equal oblique lines be drawn from the point B to the plane, they will all meet the plane in the circumference of a circle having the foot of the perpendicular for its center. It follows from this, that, if three points be taken in a plane equally distant from a point out of it, the center of the circle whose circumference passes through these points will be the foot of the perpendicular drawn from the point to the plane.

THEOREM V.

The line which joins any point of a perpendicular to a plane, with the point in which a line in the plane is intersected, at right angles, by a line through the foot of the perpendicular, will be at right angles to the line in the plane

Let AB be perpendicular to the plane ST, and AD a line through its foot at right angles to EF, a line in the plane. Connect D with any point, as B, of the perpendicular; and BD will be perpendicular to EF.



Make $DF = DE$, and join B to the points E and F. Since $DE = DF$, and the angles at D are right angles, the oblique lines, AE and AF, are equal; and, since $AE = AF$, we have, (Th. 4), $BE = BF$; therefore the line BD has two points, B and D, each equally distant from the extremities E and F of the line EF, and hence BD is perpendicular to EF at its middle point D.

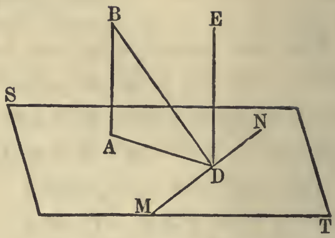
Cor. Since FD is perpendicular to the two lines AD and BD at their intersection, it is perpendicular to their plane ADB, (Th. 3).

Scholium.—The inclination of a line to a plane is measured by the angle included between the given line and the line which joins the point in which it meets the plane and the foot of the perpendicular drawn from any point of the line to the plane; thus, the angle BFA is the inclination of the line BF to the plane ST.

THEOREM VI.

If either of two parallels is perpendicular to a plane, the other is also perpendicular to the plane.

Let BA and ED be two parallels, of which one, BA, is perpendicular to the plane ST; then will the other also be perpendicular to the same plane.



The two parallels determine a plane which intersects the given plane in AD; through D draw MN perpendicular to AD; then, (Cor., Th. 5,) will MN be perpendicular to the plane BAD, and the angle MDE is therefore a right angle; but EDA is also a right angle, since BA and ED are parallel, and BAD is a right angle by hypothesis; hence, ED is perpendicular to the two lines MD and AD in the plane ST; it is therefore perpendicular to the plane, (Th. 3).

Cor. 1. The converse of this proposition is also true, that is, *if two straight lines are both perpendicular to the same plane, the lines are parallel.*

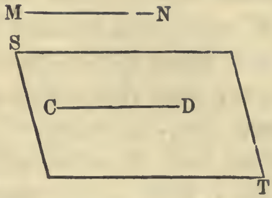
For, suppose BA and ED to be two perpendiculars; if not parallel, draw through D a parallel to BA, and this last line will be perpendicular to the plane; but ED is a perpendicular by hypothesis, and we should have two perpendiculars erected to the plane at the same point, which is impossible, (Cor. 2, Th. 3).

Cor. 2. If two lines lying in the same plane are each parallel to a third line not in the same plane, the two lines are parallel. For, pass a plane perpendicular to the third line, and it will be perpendicular to each of the others; hence they are parallel.

THEOREM VII.

A straight line is parallel to a plane, when it is parallel to a line in the plane.

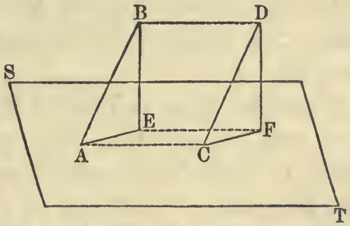
Suppose the line MN to be parallel to the line CD, in the plane ST; then will MN be parallel to the plane ST



For, CD being in the plane ST, and at the same time parallel to MN, it must be the intersection of the plane of these parallels with the plane ST; hence, if MN meet the plane ST, it must do so in the line CD, or CD produced; but MN and CD are parallel, and cannot meet; therefore MN, however far produced, can have no point in the plane ST, and hence, (Def. 5), it is parallel to this plane.

THEOREM VIII.

If two lines are parallel, they will be equally inclined to any given plane.



Let AB and CD be two parallels, and ST any plane met by them in the points A and C; then will the lines AB and CD be equally inclined to the plane ST.

For, take any distance, AB, on one of these parallels, and make $CD = AB$, and draw AC and BD. From the points B and D let fall the perpendiculars, BE and DF, on the plane; join their feet by the line EF, and draw AE and CF.

Now, since AB is equal and parallel to CD, $ABDC$ is a parallelogram, and BD is equal and parallel to AC, and BD is parallel to the plane ST, (Th. 7); and, since BE and DF are both perpendicular to this plane, they are parallel; but BD and EF are in the plane of these parallels; and as EF is in the plane ST, and BD is parallel to this plane, these two lines must be parallel and equal, and $BDFE$ is also a parallelogram Now, we have shown that BD is equal and parallel to AC, and EF equal and parallel to BD; hence, (Cor. 2, Th. 6), EF is equal and parallel to AC, and $ACFE$ is a parallelogram, and $AE = CF$. The triangles ABE and CDF have, then, the sides of the one equal to the sides of the other, each to each, and their angles are consequently equal; that is, the angle BAE is equal to the angle DCF; but these angles measure the inclination of the lines AB and CD to the plane ST, (Scholium, Th. 5).

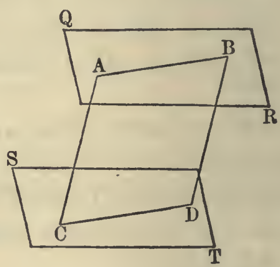
Scholium. — The converse of this proposition is not generally true; that is, straight lines equally inclined to the same plane are not necessarily parallel.

THEOREM IX.

The intersections of two parallel planes by a third plane, are parallel.

Let the planes QR and ST be intersected by the third plane, AD: then will the intersections, AB and CD, be parallel.

Since the lines AB and CD are in the same plane, if they are not parallel, they will meet if sufficiently produced; but they cannot meet out of the planes QR and ST, in which they are respectively found; therefore, any point common to the lines, must be at the same time common to the planes; and since the planes are parallel, they have no common point, and the lines, therefore, do not intersect; hence they are parallel.

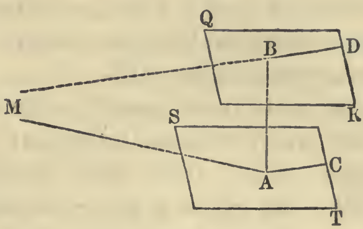


THEOREM X.

If two planes are perpendicular to the same straight line, they are parallel to each other.

Let QR and ST be two planes, perpendicular to the line AB; then will these planes be parallel.

For, if not parallel, suppose M to be a point in their line of intersection, and from this point draw lines to the extremities of the perpendicular AB, thus forming a triangle, MAB. Now, since the line AB is perpendicular to both planes, it is perpendicular to each of the lines MA and MB, drawn through its feet in the planes, (Def. 2); hence, the triangle has two right angles, which is impossible; the planes cannot therefore meet in any point as M, and are consequently parallel.

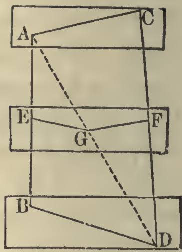


Cor. Conversely: *The straight line which is perpendicular to one of two parallel planes, is also perpendicular to the other.* For, if AB be perpendicular to the plane QR, draw in the other plane, through the point in which the perpendicular meets it, any line, as AC. The plane of the lines AB and AC will intersect the plane QR in the line BD; and since the planes are parallel by hypothesis, the lines AC and BD must be parallel, (Th. 9); but the angle DBA is a right angle; hence, BAC must be a right angle, and the line BA is perpendicular to any line whatever drawn in the plane through the point A; BA is therefore perpendicular to the plane ST.

THEOREM XI.

If two straight lines be drawn in any direction through parallel planes, the planes will cut the lines proportionally.

Conceive three planes to be parallel, as represented in the figure, and take any points, A and B, in the first and third planes, and draw AB, the line passing through the second plane at E.



Also, take any other two points, as C and D, in the first and third planes, and draw CD, the line passing through the second plane at F.

Join the two lines by the diagonal AD, which passes through the second plane at G. Draw BD, EG, GF, and AC. We are now to prove that,

$$AE : EB :: CF : FD.$$

For the sake of brevity, put $AG = X$, and $GD = Y$.

As the planes are parallel, BD is parallel to EG; from the two triangles ABD and AEG, we have, (Th. 17, B. II);

$$AE : EB :: X : Y.$$

Also, as the planes are parallel, GF is parallel to AC, and we have,

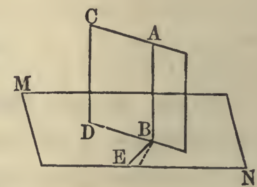
$$CF : FD :: X : Y.$$

By comparing the proportions, and applying Th. 6, B. II, we have

$$AE : EB :: CF : FD.$$

THEOREM XII.

If a straight line is perpendicular to a plane, all planes passing through that line will be perpendicular to the plane.



Let MN be a plane, and AB a perpendicular to it. Let BC be any other plane, passing through AB; this plane will be perpendicular to MN.

Let BD be the common intersection of the two planes, and from the point B, draw in MN BE at right angles to DB.

Then, as AB is perpendicular to the plane MN, it is perpendicular to every line in that plane, passing through

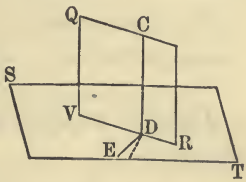
B; (Def. 2,); therefore, ABE is a right angle. But the angle ABE, (Def. 3), measures the inclination of the two planes; therefore, the plane CB is perpendicular to the plane MN; and thus we can show that any other plane, passing through AB, will be perpendicular to MN.

Hence the theorem.

THEOREM XIII.

If two planes are perpendicular to each other, and a line be drawn in one of them perpendicular to their common intersection, it will be perpendicular to the other plane.

Let the two planes, QR and ST, be perpendicular to each other, and draw in QR the line CD at right angles to their common intersection, RV; then will this line be perpendicular to the plane ST.

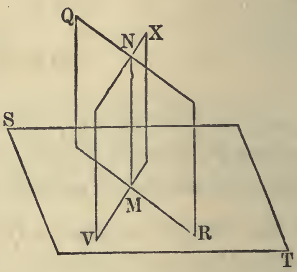


In the plane ST draw ED, perpendicular to VR at the point D. Then, since the planes QR and ST are perpendicular to each other, the angle CDE is a right angle, and CD is perpendicular to the two lines, ED and VR, passing through its foot in the plane ST. CD is therefore perpendicular to the plane ST, (Th. 3).

Cor. Conversely: if we erect a perpendicular to the plane ST, at any point, D, of its intersection with the plane QR, this perpendicular will lie in the plane QR. For, if it be not in this plane, we can draw in the plane the line CD, at right angles to VR; and, from what has been shown above, CD is perpendicular to the plane ST, and we should thus have two perpendiculars erected to the plane, ST, at the same point, which is impossible, (Cor. 2, Th. 3).

THEOREM XIV.

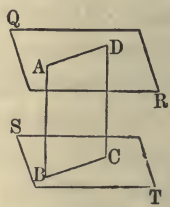
The common intersection of two planes, both of which are perpendicular to a third plane, will also be perpendicular to the third plane.



Let MN be the common intersection of the two planes, QR and VX, both of which are perpendicular to the plane ST; then will MN be perpendicular to the plane ST. For, if we erect a perpendicular to the plane ST, at the point M, it will lie in both planes at the same time, (Cor. Th. 13); and this perpendicular must therefore be their intersection. Hence the theorem.

THEOREM XV.

Parallel straight lines included between parallel planes, are equal.



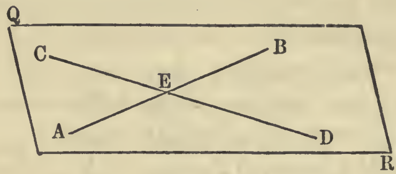
Let AB and DC be two parallel lines, included by the two parallel planes, QR and ST; then will $AB = DC$.

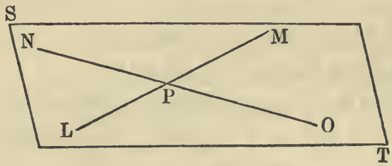
For, the plane AC, of the parallel lines, intersects the planes, QR and ST, in the parallel lines, AD and BC, (Th. 9); hence $ABCD$ is a parallelogram, and its opposite sides, AB and DC, are equal.

Cor. It follows from this proposition, that parallel planes are everywhere equally distant; for, two perpendiculars drawn at pleasure between the two planes are parallel lines, (Cor. 1, Th. 6), and hence are equal; but these perpendiculars measure the distance between the planes.

THEOREM XVI.

Two planes are parallel when two lines not parallel, lying in the one, are respectively parallel to two lines lying in the other.





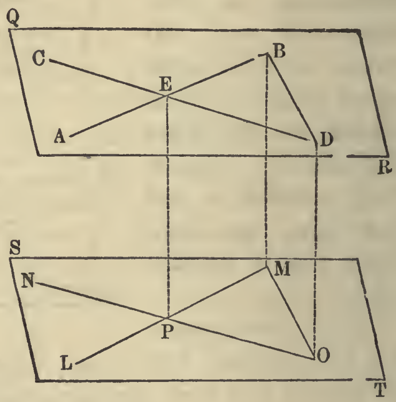
Let QR and ST be two planes, the first containing the two lines AB and CD which intersect each other at E, and the second the two lines LM and NO, respectively parallel to AB and CD; then will these planes be parallel.

For, if the two planes are not parallel, they must intersect when sufficiently produced; and their common section lying in both planes at the same time, would be a line of the plane QR. Now, the lines AB and CD intersect each other by hypothesis; hence one or both of them must meet the common section of the two planes. Suppose AB to meet this common section; then, since AB and LM are parallel, they determine a plane, and AB cannot meet the plane ST in a point out of the line LM; but AB and LM being parallel, have no common point. Hence, neither AB nor CD can meet the common section of the two planes; that is, they have no common section, and are therefore parallel.

Cor. Since two lines which intersect each other, determine a plane, it follows from this proposition, that *the plane of two intersecting lines is parallel to the plane of two other intersecting lines respectively parallel to the first lines*

THEOREM XVII.

When two intersecting lines are respectively parallel to two other intersecting lines lying in a different plane, the angles formed by the last two lines will be equal to those formed by the first two, each to each, and the planes of the angles will be parallel.



Let *QR* be the plane of the two lines *AB* and *CD*, which intersect each other at the point *E*, and *ST* the plane of the two lines *LM* and *NO*, respectively parallel to *AB* and *CD*; then will the $\angle\, BED = \angle\, MPO$, and $\angle\, BEC = \angle\, MPN$, etc., and the planes *QR* and *ST* will be parallel.

That the plane of one set of angles is parallel to that of the other, follows from the Corollary to Theorem 16; we have then only to show that the angles are equal, each to each.

Take any points, *B* and *D*, on the lines *AB* and *CD*, and draw *BD*. Lay off *PM*, equal to and in the same direction with *EB*, and *PO*, equal to and in the same direction with *ED*, and draw *MO*. Now, since the planes *QR* and *ST* are parallel, and *ED* is equal and parallel to *PO*, *EDOP* is a parallelogram, and *DO* is equal and par allel to *EP*. For the same reason, *BM* is equal and parallel to *EP*; therefore, *BDOM* is a parallelogram, and *MO* is equal and parallel to *BD*. Hence the △'s, *EBD* and *PMO*, have the sides of the one equal to the sides of the other, each to each; they are therefore equal, and the $\angle MPO =$ the $\angle BED$. In the same manner it can be proved that $\angle BEC = \angle MPN$, etc.

Cor. 1. The plane of the parallels AB and LM is intersected by the plane of the parallels CD and NO, in the line EP. Now, EB and ED are the intersections of these two planes with the plane QR, and PM and PO are the intersections of the same planes with the parallel plane ST. It has just been proved that the $\angle BED = \angle MPO$. Hence, *if the diedral angle formed by two planes, be cut by two parallel planes, the intersections of the faces of the diedral angle with one of these planes will include an angle equal to that included by the intersections of the faces with the other plane.*

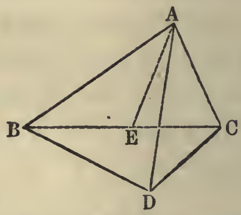
Cor. 2. *The opposite triangles formed by joining the corresponding extremities of three equal and parallel straight lines lying in different planes, will be equal and the planes of the triangles will be parallel.*

Let EP, BM, and DO, be three equal and parallel straight lines lying in different planes. By joining their corresponding extremities, we have the triangles EBD and PMO. Now, since EP and BM are equal and parallel, $EBMP$ is a parallelogram, and EB is equal and parallel to PM; in the same manner, we show that ED is equal and parallel to PO, and BD to MO; hence the triangles are equal, having the three sides of the one, respectively, equal to the three sides of the other. That their planes are parallel, follows from Cor., Theorem 16.

THEOREM XVIII.

Any one of the three plane angles bounding a triedral angle, is less than the sum of the other two.

Let A be the vertex of a solid angle, bounded by the three plane angles, BAC, BAD, and DAC; then will any one of these three angles be less than the sum of the other two. To establish this proposition, we have only to compare the greatest of the three angles with the sum of the other two.



Suppose, then, BAC to be the greatest angle, and draw in its plane the line AE, making the angle CAE equal to the angle CAD. On AE, take any point, E, and through it draw the line CEB. Take AD, equal to AE, and draw BD and DC.

Now, the two triangles, CAD and CAE, having two sides and the included angle of the one equal to the two sides and included angle of the other, each to each, are equal, and $CE = CD$; but in the triangle, BDC, $BC < BD + DC$. Taking EC from the first member of this inequality, and its equal, DC, from the second, we have, $BE < BD$. In the triangles, BAE and BAD, BA is common, and $AE = AD$ by construction; but the third side, BD, in the one, is greater than the third side, BE, in the other; hence, the angle BAD is greater than the angle BAE, (Th. 22, B. I); that is, $\llcorner BAE < \llcorner BAD$; adding the $\llcorner EAC$ to the first member of this inequality, and its equal, the $\llcorner DAC$, to the other, we have

$$\llcorner BAE + \llcorner EAC < \llcorner BAD + \llcorner DAC.$$

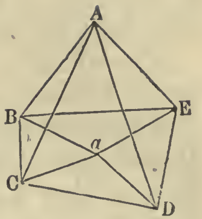
And, as the $\llcorner BAC$ is made up of the angles BAE and EAC, we have, as enunciated,

$$\llcorner BAC < \llcorner BAD + \llcorner DAC.$$

THEOREM XIX.

The sum of the plane angles forming any solid angle, is always less than four right angles.

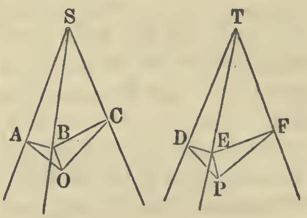
Let the planes which form the solid angle at A, be cut by another plane, which we may call the plane of the base, $BCDE$. Take any point, a, in this plane, and draw aB, aC, aD, aE, etc., thus making as many triangles on



the plane of the base as there are triangular planes forming the solid angle A. Now, since the sum of the angles of every $\triangle$ is two right angles, the sum of all the angles of the $\triangle$'s which have their vertex in A, is equal to the sum of all angles of the $\triangle$'s which have their vertex in a. But, the angles $BCA + ACD$, are, together, greater than the angles $BCa + aCD$, or BCD, by the last proposition. That is, the sum of all the angles at the bases of the $\triangle$'s which have their vertex in A, is greater than the sum of all the angles at the bases of the $\triangle$'s which have their vertex in a. Therefore, the sum of all the angles at a is greater than the sum of all the angles at A; but the sum of all the angles at a is equal to four right angles; therefore, the sum of all the angles at A is less than four right angles.

THEOREM XX.

If two solid angles are formed by three plane angles respectively equal to each other, the planes which contain the equal angles will be equally inclined to each other.



Let the $\angle ASC$ = the $\angle DTF$, the $\angle ASB$ = the $\angle DTE$, and the $\angle BSC$ = the $\angle ETF$; then will the inclination of the planes, ASC, ASB, be equal to that of the planes, DTF, DTE.

Having taken SB at pleasure, draw BO perpendicular to the plane ASC; from the point O, at which that perpendicular meets the plane, draw OA and OC, perpendicular to SA and SC; draw AB and BC; next take $TE = SB$, and draw EP perpendicular to the plane DTF; from the point P, draw PD and PF, perpendicular to TD and TF; lastly, draw DE and EF.

The triangle SAB, is right-angled at A, and the triangle TDE, at D, (Th. 5); and since the $\angle ASB =$ the $\angle DTE$, we have $\angle SBA = \angle TED$; likewise, $SB = TE$; therefore, the triangle SAB is equal to the triangle TDE; hence, $SA = TD$, and $AB = DE$. In like manner it may be shown that $SC = TF$, and $BC = EF$. That granted, the quadrilateral $SAOC$ is equal to the quadrilateral $TDPF$; for, place the angle ASC upon its equal, DTF, and because $SA = TD$, and $SC = TF$, the point A will fall on D, and the point C on F; and, at the same time, AO, which is perpendicular to SA, will fall on PD, which is perpendicular to TD, and, in like manner, OC on PF; wherefore, the point O will fall on the point P, and AO will be equal to DP. But the triangles, AOB, DPE, are right angled at O and P; the hypotenuse $AB = DE$, and the side $AO = DP$; hence, those triangles are equal, (Cor, Th. 39, B. I), and $\angle OAB = \angle PDE$. The angle OAB is the inclination of the two planes, ASB, ASC; the angle PDE is that of the two planes, DTE, DTF; consequently, those two inclinations are equal to each other.

Hence the theorem.

Scholium 1. — The angles which form the solid angles at S and T, may be of such relative magnitudes, that the perpendiculars, BO and EP, may not fall within the bases, ASC and DTF; but they will always either fall on the bases, or on the planes of the bases produced, and O will have the same relative situation to A, S, and C, as P has to D, T, and F. In case that O and P fall on the planes of the bases produced, the angles BCO and EFP, would be obtuse angles; but the demonstration of the problem would not be varied in the least.

Scholium 2. — If the plane angles bounding one of the triedral angles be equal to those of the other, each to each, and also be similarly arranged about the triedral angles, these solid angles will be absolutely equal. For it was shown, in the course of the above demonstration, that the quadrilaterals, $SAOC$ and $TDPF$, were equal; and on being applied, the point O falls on the point P; and since the triangles AOB and DPE are equal, the perpendiculars OB and PE are also equal. Now, because the plane angles are like arranged about the triedral angles, these perpendiculars lie in the same direction; hence the point B will fall on the point E, and the solid angles will exactly coincide.

Scholium 3.—When the planes of the equal angles are not like disposed about the triedral angles, it would not be possible to make these triedral angles coincide; and still it would be true that the planes of the equal angles are equally inclined to each other. Hence, these triedral angles have the plane and diedral angles of the one, equal to the plane and diedral angles of the other, each to each, without having of themselves that absolute equality which admits of superposition. Magnitudes which are thus equal in all their component parts, but will not coincide, when applied the one to the other, are said to be *symmetrically equal.* Thus, two triedral angles, bounded by plane angles equal each to each, but not like placed, are ***symmetrical triedral angles.***

BOOK VII.

SOLID GEOMETRY.

DEFINITIONS.

1. A **Polyedron** is a solid, or volume, bounded on all sides by planes. The bounding planes are called the *faces* of the polyedron, and their intersections are its *edges.*

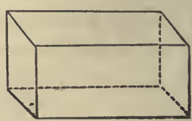
2. A **Prism** is a polyedron, having two of its faces, called *bases,* equal polygons, whose planes and homologous sides are parallel. The other, or *lateral faces,* are parallelograms, and constitute the *convex surface* of the prism.

The bases of a prism are distinguished by the terms, *upper* and *lower;* and the *altitude* of the prism is the perpendicular distance between its bases.

Prisms are denominated *triangular, quadrangular, pentangular, etc.,* according as their bases are *triangles, quadrilaterals, pentagons, etc.*

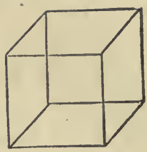
3. A **Right Prism** is one in which the planes of the lateral faces are perpendicular to the planes of the bases.

4. A **Parallelopipedon** is a prism whose bases are parallelograms.



5. A **Rectangular Parallelopipedon** is a right parallelopipedon, with rectangular bases.

6. A **Cube** or **Hexaedron** is a rectangular parallelopipedon, whose faces are all equal squares.

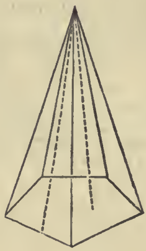


7. A **Diagonal of a Polyedron** is a straight line joining the vertices of two solid angles not adjacent.

8. **Similar Polyedrons** are those which are bounded by the same number of similar polygons like placed, and whose homologous solid angles are equal.

Similar parts, whether faces, edges, diagonals, or angles, similarly placed in similar polyedrons, are termed *homologous.*

9. A **Pyramid** is a polyedron, having for one of its faces, called the *base,* any polygon whatever, and for its other faces triangles having a common vertex, the sides opposite which, in the several triangles, being the sides of the base of the pyramid.



10. The **Vertex** of a pyramid is the common vertex of the triangular faces.

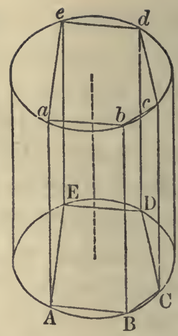
11. The **Altitude** of a pyramid is the perpendicular distance from its vertex to the plane of its base.

12. A **Right Pyramid** is one whose base is a regular polygon, and whose vertex is in the perpendicular to the base at its center. This perpendicular is called the *axis* of the pyramid.

13. The **Slant Height** of a right pyramid is the perpendicular distance from the vertex to one of the sides of the base.

14. The **Frustum of a Pyramid** is a portion of the pyramid included between its base and a section made by a plane parallel to the base.

Pyramids, like prisms, are named from the forms of their bases.



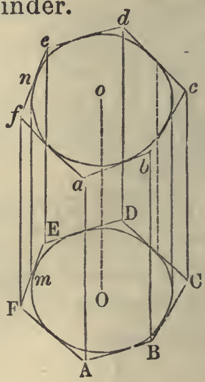
15. A **Cylinder** is a body, having for its ends, or bases, two equal circles, the planes of which are perpendicular to the line joining their centers; the remainder of its surface may be conceived as formed by the motion of a line, which constantly touches the circumferences of the bases, while it remains parallel to the line which joins their centers.

We may otherwise define the cylinder as a body generated by the revolution of a rectangle about one of its sides as an immovable axis.

The sides of the rectangle perpendicular to the axis generate the *bases* of the cylinder; and the side opposite the axis generates its *convex surface.* The line joining the centers of the bases of the cylinder is its *axis*, and is also its *altitude.*

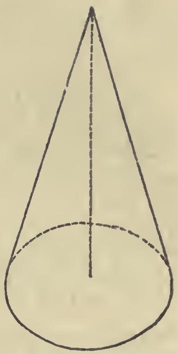
If, within the base of a cylinder, any polygon be inscribed, and on it, as a base, a right prism be constructed, having for its altitude that of the cylinder, such prism is said to be *inscribed in the cylinder*, and the cylinder is said to *circumscribe the prism.*

Thus, in the last figure, *ABCDEc* is an inscribed prism, and it is plain that all its lateral edges are contained in the convex surface of the cylinder.



If, about the base of a cylinder, any polygon be circumscribed, and on it, as a base, a right prism be constructed, having for its altitude that of the cylinder, such prism is said to be *circumscribed about the cylinder*, and the cylinder is said to be *inscribed in the prism.*

Thus, *ABCDEFc* is a circumscribed prism; and it is plain that



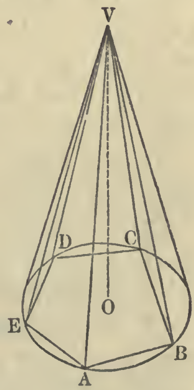
the line, *mn*, which joins the points of tangency of the sides, *EF* and *ef*, with the circumferences of the bases of the cylinder, is common to the convex surfaces of the cylinder and prism.

16. A **Cone** is a body bounded by a circle and the surface generated by the motion of a straight line, which constantly passes through a point in the perpendicular to the plane of the circle at its center, and the different points in its circumference.

The cone may be otherwise defined as a body generated by the revolution of a right-angled triangle about one of its sides as an immovable axis. The other side of the triangle will generate the *base* of the cone, while the hypotenuse generates the *convex surface.*

The side about which the generating triangle revolves is the *axis* of the cone, and is at the same time its *altitude.*

If, within the base of the cone, any polygon be inscribed, and on it, as a base, a pyramid be constructed, having for its vertex that of the cone, such pyramid is said to be *inscribed in the cone*, and the cone is said to *circumscribe the pyramid.*



Thus, in the accompanying figure, *V—ABCDE*, is an inscribed pyramid, and it is plain that all its lateral edges are contained in the convex surface of the cone.

If, about the base of a cone, any polygon be circumscribed, and on it, as a base, a pyramid be constructed, having for its vertex that of the cone, such pyramid is said to be *circumscribed about the cone*, and the cone is said to be *inscribed in the pyramid.*

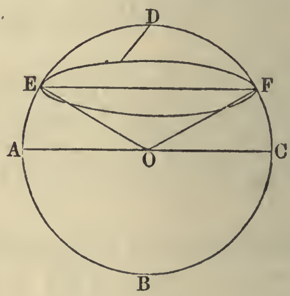
17. The **Frustum of a Cone** is the portion of the cone that is included between its base and a section made by a plane parallel to the base.

18. **Similar Cylinders**, and also **Similar Cones**, are such as have their axes proportional to the radii of their bases.

19. A **Sphere** is a body bounded by one uniformly-curved surface, all the points of which are at the same distance from a certain point within, called the *center*.

We may otherwise define the sphere as a body generated by the revolution of a semicircle about its diameter as an immovable axis.

20. A **Spherical Sector** is that portion of a sphere which is included between the surfaces of two cones having a common axis, and their vertices at the center of the sphere. Or, it is that portion of the sphere which is generated by a sector of the generating semicircle.



21. The **Radius of a Sphere** is a straight line drawn from the center to any point in the surface; and the *diameter* is a straight line drawn through the center, and limited on both sides by the surface.

All the diameters of a sphere are equal, each being twice the radius.

22. A **Tangent Plane** to a sphere is one which has a single point in the surface of the sphere, all the others being without it.

23. A **Secant Plane** to a sphere is one which has more than one point in the surface of the sphere, and lies partly within and partly without it.

Assuming, what will presently be proved, that the intersection of a sphere by a plane is a circle.

24. A **Small Circle** of a sphere is one whose plane does not pass through its center; and

25. A **Great Circle** of a sphere is one whose plane passes through the center of the sphere.

26. A **Zone** of a sphere is the portion of its surface included between the circumferences of any two of its parallel circles, called the *bases* of the zone. When the plane of one of these circles becomes tangent to the sphere, the zone has a single base.

27. A **Spherical Segment** is a portion of the volume of a sphere included between any two of its parallel circles, called the *bases* of the segment.

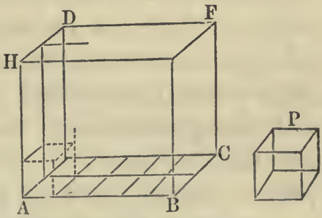
The altitude of a zone, or of a segment, of a sphere, is the perpendicular distance between the planes of its bases.

28. The area of a surface is measured by the product of its *length* and *breadth*, and these dimensions are always conceived to be exactly at right angles to each other.

29. In a similar manner, solids are measured by the product of their *length*, *breadth*, and *height*, when all their dimensions are at right angles to each other.

The product of the length and breadth of a solid, is the measure of the *surface* of its base.

Let P, in the annexed figure, represent the measuring unit, and AF the rectangular solid to be measured.



A side of P is one unit in length, one in breadth, and one in height; one inch, one foot, one yard, or any other unit that may be taken.

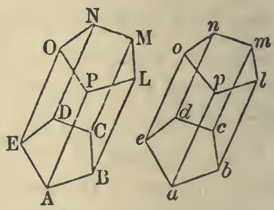
Then, $1 \times 1 \times 1 = 1$, the *unit cube*.

Now, if the base of the solid, AC, is, as here represented, 5 units in length and 2 in breadth, it is obvious that $(5 \times 2 = 10)$, 10 units, each equal to P, can be placed on the base of AC, and no more; and as each of these units will occupy a unit of altitude, therefore, 2 *units* of altitude will contain 20 solid units, 3 units of altitude, 30 solid units, and so on; or, in general terms, *the number of square units in the base multiplied by the linear units in perpendicular altitude, will give the solid units in any rectangular solid.*

THEOREM I.

If the three plane faces bounding a solid angle of one prism be equal to the three plane faces bounding a solid angle of another, each to each, and similarly disposed, the prisms will be equal.

Suppose A and a to be the vertices of two solid angles, bounded by equal and similarly placed faces; then will the prisms, $ABCDE$—N and $abcde$—n, be equal.



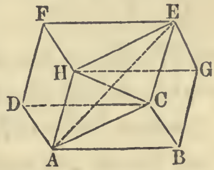
For, if we place the base, $abcde$, upon its equal, the base $ABCDE$, they will coincide; and since the solid angles, whose vertices are A and a, are equal, the lines ab, ae, and ap, respectively coincide with AB, AE, and AP; but the faces, al and ao, of the one prism, are equal, each to each, to the faces, AL and AO, of the other; therefore pl and po coincide with PL and PO, and the upper bases of the prisms also coincide: hence, not only the bases, but all the lateral faces of the two prisms coincide, and the prisms are equal.

Cor. If the two prisms are right, and have equal bases and altitudes, they are equal. For, in this case, the rectangular faces, al and ao, of the one, are respectively equal to the rectangular faces, AL and AO, of the other; and hence the three faces bounding a triedral angle in the one, are equal and like placed, to the faces bounding a triedral angle in the other

THEOREM II.

The opposite faces of any parallelopipedon are equal, and their planes are parallel.

Let $ABCD—E$ be any parallelopipedon; then will its opposite faces be equal, and their planes will be parallel.



The bases $ABCD$ and $FEGH$ are equal, and their planes are parallel, by definitions 2 and 4 of this Book; it remains for us, therefore, only to show that any two of the opposite lateral faces are equal and parallel.

Since all the faces of the parallelopipedon are parallelograms, AB is equal and parallel to DC, and AH is also equal and parallel to DF; hence the angles HAB and FDC are equal, and their planes are parallel, (Th. 17, B. VI), and the two parallelograms, $HABG$ and $FDCE$, having two adjacent sides and the included angle of the one equal to the two adjacent sides and included angle of the other, are equal.

Cor. 1 Hence, of the six faces of the parallelopipedon, any two lying opposite may be taken as the bases.

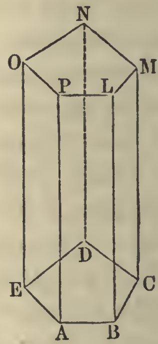
Cor. 2. The four diagonals of a parallelopipedon mutually bisect each other. For, if we draw AC and HE, we shall form the parallelogram $ACEH$, of which the diagonals are AE and HC, and these diagonals are at the same time diagonals of the parallelopipedon; but the diagonals of a parallelogram mutually bisect each other. Now, if the diagonal FB be drawn, it and HC will bisect each other, since they are diagonals of the parallelogram $FHBC$. In like manner we can show that if DG be drawn, it will be bisected by AE. Hence, the four diagonals have a common point within the parallelopipedon.

SCHOLIUM.—It is seen at once that the six faces of a parallelopipedon intersect each other in twelve edges, four of which are equal to HA, four to AB, and four to AD. Now, we may conceive the parallelopipedon to be bounded by the planes determined by the three lines

AH, AB, and AD, and the three planes passed through the extremities, H, B, and D, of these lines, parallel to the first three planes.

THEOREM III.

The convex surface of a right prism is measured by the perimeter of its base multiplied by its altitude.



Let $ABCDE - N$ be a right prism, of which AP is the altitude; then will its convex surface be measured by

$$(AB + BC + CD + DE + EA) \times AP.$$

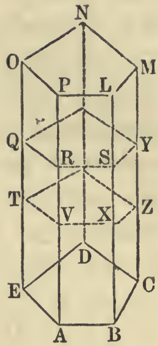
For, its convex surface is made up of the rectangles AL, BM, CN, etc., and each rectangle is measured by the product of its base by its altitude; but the altitude of each rectangle is equal to AP, the altitude of the prism; hence the convex surface of the prism is measured by the product of the sum of the bases of the rectangles, or the perimeter of the base of the prism, by the common altitude, AP.

Cor. Right prisms will have equivalent convex surfaces, when the products of the perimeters of their bases by their altitudes are respectively equal; and, generally, their convex surfaces will be to each other as the products of the perimeters of their bases by their altitudes. Hence, if the altitudes are equal, their convex surfaces will be as the perimeters of their bases; and if the perimeters of their bases are equal, their convex surfaces will be as their altitudes.

THEOREM IV.

The two sections of a prism made by parallel planes between its bases are equal polygons.

Let the prism $ABCDE - N$ be cut between its bases by two parallel planes, making the sections QRS, etc., and TVX, etc.; then will these sections be equal polygons.

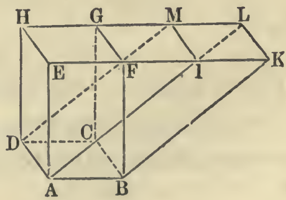


For, since the secant planes are parallel, their intersections, QR and TV, by the plane of the face $EAPO$ are parallel, (Th. 9, B. VI); and being included between the parallel lines, AP and EO, they are also equal. In the same manner we may prove that RS is equal and parallel to VX, and so on for the intersections of the secant planes by the other faces of the prism. Hence, these polygonal sections have the sides of the one equal to the sides of the other, each to each. The angles QRS and TVX are equal, because their sides are parallel and lie in the same direction; and in like manner we prove $\angle RSY = \angle VXZ$, and so on for the other corresponding angles of the polygons. Therefore, these polygons are both mutually equilateral and mutually equiangular, and consequently are equal.

Cor. A section of a prism made by a plane parallel to the base of the prism, is a polygon equal to the base.

THEOREM V.

Two parallelopipedons, the one rectangular and the other oblique, will be equal in volume when, having the same base and altitude, two opposite lateral faces of the one are in the planes of the corresponding lateral faces of the other.



Designating the parallelopipedons by their opposite diagonal letters, let AG be the rectangular, and AL the oblique, parallelopipedon, having the same base, AC, and the same altitude, namely, the perpendicular distance between the parallel planes, AC and EL. Also let the face, AK, be in the plane of the face, AF, and the face, DL, in the plane of the face, DG. We are now to prove that the oblique parallelopipedon is equivalent to the rectangular parallelopipedon.

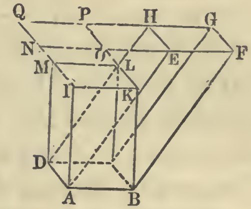
As the faces, AF and AK, are in the same plane, and the parallelopipedons have the same altitude, EFK is a straight line, and $EF = IK$, because each is equal to AB. If from the whole line, EK, we take EF, and then from the same line we take $IK = EF$, we shall have the remainders, EI and FK, equal; and since AE and BF are parallel, $\llcorner AEI = \llcorner BFK$; hence the $\triangle$'s, AEI and BFK, are equal. Since HE and MI are both parallel to DA, they are parallel to each other, and $EIMH$ is a parallelogram; for like reasons, $FKLG$ is a parallelogram, and these parallelograms are equal, because two adjacent sides and the included angle of the one are equal to two adjacent sides and the included angle of the other. The parallelograms, DE and CF, being the opposite faces of the parallelopipedon, AG, are equal. Hence, the three plane faces bounding the triedral angle, E, of the triangular prism, $EAI—H$, are equal, each to each, and like placed, to the three plane faces bounding the triedral angle F, of the triangular prism, $FBK—G$, and these prisms are therefore equal, (Th. 1). Now, if from the whole solid, $EABK—H$, we take the prism, $EAI—H$, there will remain the parallelopipedon, AL; and, if from the same solid, we take the prism, $FBK—G$, there will remain the rectangular parallelopipedon, AG. Therefore, the oblique and the rectangular parallelopipedons are equivalent.

Cor. The volume of the rectangular parallelopipedon, AG, is measured by the base, $ABCD$, multiplied by the altitude, AE, (Def. 29); consequently, the oblique parallelopipedon is measured by the product of the same base by the same altitude.

Scholium.—If neither of the parallelopipedons is rectangular, but they still have the same base and the same altitude, and two opposite lateral faces of the one are in the planes of the corresponding lateral faces of the other, by precisely the same reasoning we could prove the parallelopipedons equivalent. Hence, in general, *any two parallelopipedons will be equal in volume when, having the same base and altitude, two opposite lateral faces of the one are in the planes of the corresponding lateral faces of the other.*

THEOREM VI.

Two parallelopipedons having equal bases and equal altitudes, are equivalent.



Let AG and AL be two parallelopipedons, having a common lower base, and their upper bases in the same plane, HF. Then will these parallelopipedons be equivalent.

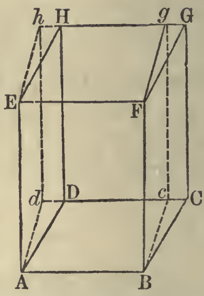
Since their upper bases are in the same plane, and the lines IM and KL are parallel, and also EF and HG, these lines will intersect, when produced, and form the parallelogram $NOPQ$, which will be equal to the common lower base of the two parallelopipedons. Now, if a third parallelopipedon be constructed, having BD for its lower base, and OQ for its upper base, it will be equivalent to the parallelopipedon AG, and also to the parallelopipedon AL, (Th. 5, Scholium); hence, the two given parallelopipedons, being each equivalent to the third parallelopipedon, are equivalent to each other.

Hence, *two parallelopipedons having equal bases, etc.*

THEOREM VII.

The volume of any parallelopipedon is measured by the product of its base and altitude, or the product of its three dimensions.

Let $ABCD-G$ be any parallelopipedon; then will its volume be expressed by the product of the area of its base and altitude.



If the parallelopipedon is oblique, we may construct on its base a right parallelopipedon, by erecting perpendiculars at the points A, B, C, and D, and making them each equal to the altitude of the given parallelopipedon; and the right parallelopipedon, thus constructed, will be equivalent to the given parallelopipedon, (Th. 6). Now, if the base, $ABCD$, is a rectangle, the new parallelopipedon will be rectangular, and measured by the product of its base and altitude, (Def. 29). But if the base is not rectangular, let fall the perpendiculars, Bc and Ad, on CD and CD produced, and take the rectangle $ABcd$ for the base of a rectangular parallelopipedon, having for its altitude that of the given parallelopipedon. We may now regard the rectangular face, $ABFE$, as the common base of the two parallelopipedons, Ag and AG; and, as they have a common base, and equal altitude, they are equivalent. Thus we have reduced the oblique parallelopipedon, first to an equivalent right parallelopipedon on the same base, and then the right to an equivalent rectangular parallelopipedon on an equivalent base, all having the same altitude. But the rectangular parallelopipedon, Ag, is measured by product of its base, $ABcd$, and its altitude; hence, the given and equivalent oblique parallelopipedon is measured by the product of its equivalent base and equal altitude.

Hence, *the volume of any parallelopipedon, etc.*

Cor. Since a parallelopipedon is measured by the product of its base by its altitude, it follows that *parallelopipedons of equivalent bases, and equal altitudes, are equivalent, or equal in volume.*

THEOREM VIII.

Parallelopipedons on the same, or equivalent bases, are to each other as their altitudes; and parallelopipedons having equal altitudes, are to each other as their bases.

Let P and p represent two parallelopipedons, whose bases are denoted by B and b, and altitudes by A and a, respectively.

Now, $P = B \times A$, and $p = b \times a$, (Th. 7).

But magnitudes are proportional to their numerical measures; that is,

$$P : p :: B \times A : b \times a.$$

If the bases of the parallelopipedons are equivalent, we have $B = b$; and if the altitudes are equal, we have $A = a$. Introducing these suppositions, in succession, in the above proportion, we get

$$P : p :: A : a,$$

and $$P : p :: B : b.$$

Hence the theorem; *Parallelopipedons on the same, etc.*

THEOREM IX.

Similar parallelopipedons are to each other as the cubes of their like dimensions.

Let P and p represent any two similar parallelopipedons, the altitude of the first being denoted by h, and the length and breadth of its base by l and n, respectively; and let h', l', and n', in order, denote the corresponding dimensions of the second.

Then we are to prove that

$$P : p :: n^3 : n'^3 :: l^3 : l'^3 :: h^3 : h'^3.$$

We have

$$P = lnh, \text{ and } p = l'n'h' \text{ (Th. 7);}$$

and by dividing the first of these equations by the second, member by member, we get

$$\frac{P}{p} = \frac{lnh}{l'n'h'};$$

which, reduced to a proportion, gives

$$P : p :: lnh : l'n'h'.$$

But, by reason of the similarity of the parallelopipedons, we have the proportions

$$l : l' :: n : n'$$
$$h : h' :: n : n';$$

we have also the identical proportion,

$$n : n' :: n : n'.$$

By the multiplication of these proportions, term by term, we get, (Th. 11, B. II),

$$lnh : l'n'h' :: n^3 : n'^3.$$

That is, $P : p :: n^3 : n'^3.$

By treating in the same manner the three proportions,

$$l : l' :: h : h'$$
$$n : n' :: h : h'$$
$$h : h' :: h : h',$$

we should obtain the proportion

$$P : p :: h^3 : h'^3;$$

and, by a like process, the three proportions,

$$h : h' :: l : l'$$
$$n : n' :: l : l'$$
$$l : l' :: l : l',$$

will give us the proportion

$$P : p :: l^3 : l'^3.$$

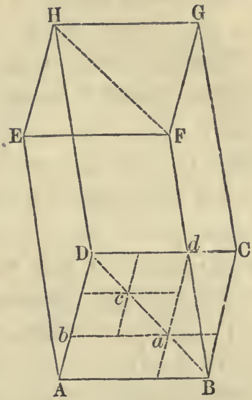
Hence the theorem; *similar parallelopipedons are to each other, etc.*

THEOREM X.

The two triangular prisms into which any parallelopipedon is divided, by a plane passing through its opposite diagonal edges, are equivalent.

Let $ABCD-F$ be a parallelopipedon, and through the diagonal edges, BF and DH, pass the plane BH, dividing the parallelopipedon into the two triangular prisms,

$ABD-E$ and $BCD-G$; then we are to prove that these prisms are equivalent. Let us divide the diagonal, BD, in which the secant plane intersects the base of the parallelopipedon, into three equal parts, a and c being the points of division. In the base, $ABCD$, construct the complementary parallelograms, aC and aA, and in the parallelogram, $badD$, construct the complementary parallelograms, cd and cb, and conceive these, together with the parallelograms, Ba, ac, cD, to be the bases of smaller parallelopipedons, having their lateral faces parallel to the lateral faces of, and their altitude equal to the altitude of, the given parallelopipedon, AG.



Now it is evident that the triangular prism, $BCD-G$, is composed of the parallelopipedons on the bases, aC and cd, and the triangular prisms, on the side of the secant plane with this prism, into which this plane divides the parallelopipedons on the bases, Ba, ac, and cD. The triangular prism, $ABD-E$, is also composed of the parallelopipedons on the bases, Aa and bc, together with the triangular prisms on the side of the secant plane with this prism, into which this plane divides the parallelopipedons on the bases, Ba, ac, and cD.

But the parallelograms, aC and aA, being complementary, are equivalent, (Th. 31, B. I); and for the same reason the parallelograms, cd and cb, are equivalent; and since parallelopipedons on equivalent bases and of equal altitudes, are equivalent, (Cor., Th. 7), we have the sum of parallelopipedons on bases aC and cd, equivalent to the sum of parallelopipedons on the bases, aA and cb. Hence, the triangular prisms, $ABD-E$ and $BCD-G$, differ in volume only by the difference which may exist between the sums of the triangular prisms on the two sides of the secant plane into which this plane divides the parallelopipedons on the bases, *Ba*, *ac*, and *cd*.

Now, if the number of equal parts into which the diagonal is divided, be indefinitely multiplied, it still holds true that the triangular prisms, *ABD—E* and *BCD—G*, differ in volume only by the difference between the sums of the triangular prisms on the two sides of the secant plane into which this plane divides the parallelopipedons constructed on the bases whose diagonals are the equal portions of the diagonal, *BD*. But in this case the sum of these parallelopipedons themselves becomes an indefinitely small part of the whole parallelopipedon, *AG*, and the difference between the parts of an indefinitely small quantity must itself be indefinitely small, or less than any assignable quantity. Therefore, the triangular prisms, *ABD—E* and *BCD—G*, differ in volume by less than any assignable volume, and are consequently equivalent.

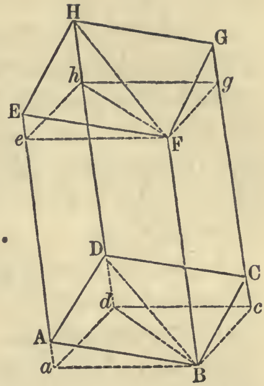
Hence the theorem; *the two triangular prisms into which, etc.*

Cor. 1. Any triangular prism, as *ABD—E*, is one half the parallelopipedon having the same triedral angle, *A*, and the same edges, *AB*, *AD*, and *AE*.

Cor. 2. Since the volume of a parallelopipedon is measured by the product of its base and altitude, and the triangular prisms into which it is divided by the diagonal plane, have bases equivalent to one half the base of the parallelopipedon, and the same altitude, it follows that, *the volume of a triangular prism is measured by the product of its base and altitude.*

The above demonstration is less direct, but is thought to be more simple, than that generally found in authors, and which is here given as a

Second Demonstration



Let $ABCD$--F be a parallelopipedon, divided by the diagonal plane, BH, passing through the edges, BF and DH; then we are to prove that the triangular prisms, ABD—E and BCD—G, thus formed, are equivalent.

Through the points B and F, pass planes perpendicular to the edge, BF, and produce the lateral faces of the parallelopipedon to intersect the plane through B; then the sections $Bcda$ and $Fghe$ are equal parallelograms. For, since the cutting planes are both perpendicular to BF, they are parallel, (Th. 10, B. VI); and because the opposite faces of a parallelopipedon are in parallel planes, (Th. 2), and the intersections of two parallel planes by a third plane are parallel, (Th. 9, B. VI), the sections, $Bcda$ and $Fghe$, are equal parallelograms, and may be taken as the bases of the right parallelopipedon, $Bcda$—h. But the diagonal plane divides the right parallelopipedon into the two equal triangular prisms, aBd—e and Bcd—g, (Th. 1). We will now compare the right prism with the oblique triangular prism on the same side of the diagonal plane.

The volume ABD—e is common to the two prisms, ABD—E and aBd—e; and the volume eFh—E, which, added to this common part, forms the oblique triangular prism, is equal to the volume aBd—A, which, added to the common part, forms the right triangular prism. For, since $ABFE$ and $aBFe$ are parallelograms, $AE = ae$, and taking away the common part Ae, we have $aA = eE$; and since $BFHD$ and $BFhd$ are parallelograms, we have $DH = dh$; and from these equals taking away the common part Dh, we have $dD = hH$. Now, if the volume eFh—H

be applied to the volume $aBd-D$, the base eFh falling on the equal base aBd, the edges eE and hH will fall upon aA and dD respectively, because they are perpendicular to the base aBd, (Cor. 2, Th. 3, B. VI), and the point E will fall upon the point A, and the point H upon the point D; hence the volume $eFh-H$ exactly coincides with the volume $aBd-D$, and the oblique triangular prism $ABD-E$ is equivalent to the right triangular prism $aBd--e$.

In the same manner, it may be proved that the oblique triangular prism, $BCDG$, is equivalent to the right triangular prism, $Bcdg$. The oblique triangular prism on either side of the diagonal plane is, therefore, equivalent to the corresponding right triangular prism; and, as the two right triangular prisms are equal, the oblique triangular prisms are equivalent.

Hence the theorem; *the two triangular prisms, etc.*

THEOREM XI.

The volume of any prism whatever is measured by the product of the area of its base and altitude.

For, by passing planes through the homologous diagonals of the upper and lower bases of the prism, it will be divided into a number of triangular prisms, each of which is measured by the product of the area of its base and altitude. Now, as these triangular prisms all have, for their common altitude, the altitude of the given prism, when we add the measures of the triangular prisms, to get that of the whole prism, we shall have, for this measure, the common altitude multiplied by the sum of the areas of the bases of the triangular prisms: that is, the product of the area of the polygonal base and the altitude of the prism.

Hence the theorem; *the volume of any prism, etc.*

Cor. If A denote the area of the base, and H the altitude of a prism, its volume will be expressed by $A \times H$. Calling this volume V, we have

$$V = A \times H.$$

Denoting by A', H', and V', in order, the area of the base, altitude, and volume of another prism, we have

$$V' = A' \times H'.$$

Dividing the first of these equations by the second, member by member, we have

$$\frac{V}{V'} = \frac{A \times H}{A' \times H'},$$

which gives the proportion,

$$V : V' :: A \times H : A' \times H'.$$

If the bases are equivalent, this proportion becomes

$$V : V' :: H : H';$$

and if the altitudes are equal, it reduces to

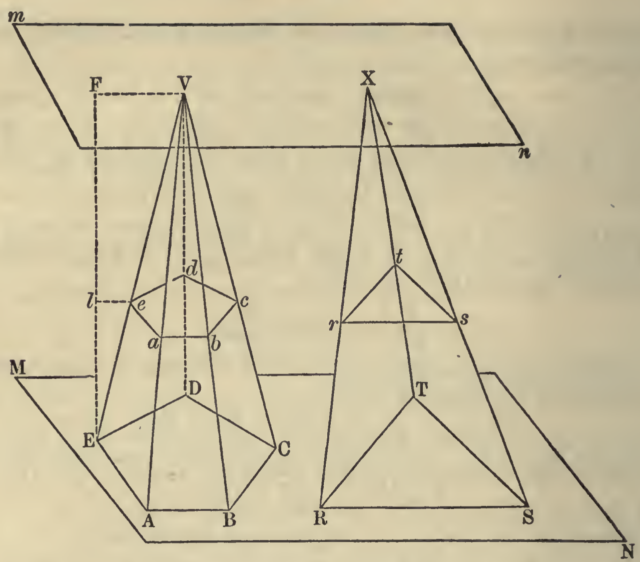
$$V : V' :: A : A'.$$

Hence, *prisms of equivalent bases are to each other as their altitudes; and prisms of equal altitudes are to each other as their bases.*

THEOREM XII.

A plane passed through a pyramid parallel to its base, divides its edges and altitude proportionally, and makes a section, which is a polygon similar to the base.

Let $ABCDE—V$ be any pyramid, whose base is in the plane, MN, and vertex in the parallel plane, mn; and let a plane be passed through the pyramid, parallel to its base, cutting its edges at the points, a b, c, d, e, and the altitude, EF, at the point l. By joining the points, a, b, c, etc., we have the polygon formed by the intersection of the plane and the sides of the pyramid. Now, we are to prove that the edges, VA, VB, etc., and the altitude, FE, are divided proportionally at the points, a, b, etc., and l; and that the polygon, a, b, c, d, e, is similar to the base of the pyramid.



Since the cutting plane is parallel to the base of the pyramid, ab is parallel to AB, (Th. 9, B. VI); for the same reason, bc is parallel to BC, cd to CD, etc. Now, in the triangle VAB, because ab is parallel to the base AB, we have, (Th. 17, B. II), the proportion,

$$VA : Va :: VB : Vb.$$

In like manner, it may be shown that

$$VB : Vb :: VC : Vc,$$

and so on for the other lateral edges of the pyramid. F being the point in which the perpendicular from E pierces the plane mn, and l the point in which the parallel secant plane cuts the perpendicular, if we join the points F and V, and also the points l and e by straight lines, we have in the triangle EFV, the line le parallel to the base FV; hence the proportion

$$VE : Ve :: FE : Fl.$$

Therefore, the plane passed through the pyramid parallel to its base, divides the altitude into parts which have to each other the same ratio as the parts into which it divides the edges.

Again, since ab is parallel to AB, and bc to BC, the angle abc is equal to the angle ABC, (Th. 17, B. VI.); in the same manner we may show that each angle in the polygon, $abcde$, is equal to the corresponding angle in the polygon, $ABCDE$; therefore these polygons are mutually equiangular. But, because the triangles VBA and Vba are similar, their homologous sides give the proportion

$$Vb : VB :: ab : AB;$$

and because the triangles Vbc and VBC are similar, we also have the proportion

$$Vb : VB :: bc : BC.$$

Since the first couplets in these two proportions are the same, the second couplets are proportional, and give

$$ab : AB :: bc : BC.$$

By a like process, we can prove that

$$bc : BC :: cd : CD,$$

and that $$cd : CD :: de : DE,$$

and so on, for the other homologous sides of the two polygons.

Hence, the two polygons are not only mutually equiangular, but the sides about the equal angles taken in the same order are proportional, and the polygons are therefore similar, (Def. 16, B. II).

Hence the theorem; *a plane passed through a pyramid, etc.*

Cor. 1. Since the areas of similar polygons are to each other as the squares of their homologous sides, (Th. 22, B. II), we have

$$\text{area } abcde : \text{area } ABCDE : \overline{ab}^2 : \overline{AB}^2.$$

But, $$ab : AB :: Va : VA :: Fl : FE;$$

hence, $$\overline{ab}^2 : \overline{AB}^2 :: \overline{Fl}^2 : \overline{FE}^2:$$

therefore, $$\text{area } abcde : \text{area } ABCDE : \overline{Fl}^2 : \overline{FE}^2.$$

That is, *the area of a section parallel to the base of a pyramid, is to the area of the base, as the square of the perpendicular distance from the vertex of the pyramid to the section, is to the square of the altitude of the pyramid.*

Cor. 2. Let V—$ABCDE$ and X—RST be two pyramids, having their bases in the plane MN, and their vertices in the parallel plane mn; and suppose a plane to be passed through the two pyramids parallel to the common plane of their bases, making in the one the section $abcde$, and in the other the section rst.

Now, area $ABCDE$: area $abcde :: \overline{AB}^2 : \overline{ab}^2$, (Th. 22, B. II),

and " RST : " $rst :: \overline{RS}^2 : \overline{rs}^2$.

But, $AB : ab :: VB : Vb$,

and $RS : rs :: XR : Xr$.

Because the plane which makes the sections is parallel to the planes MN and mn, we have, (Th. 11, B. VI),

$$VB : Vb :: XR : Xr;$$

therefore, (Cor. 2, Th. 6, B. II), $AB : ab :: RS : rs$.

By squaring, $\overline{AB}^2 : \overline{ab}^2 : \overline{RS}^2 : \overline{rs}^2$;

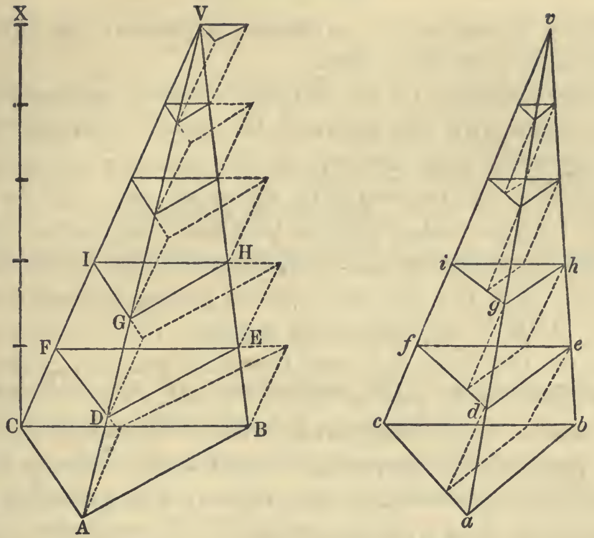
hence, area $ABCDE$: area $abcde$ :: area RST : area rst.

That is, *if two pyramids having equal altitudes, and their bases in the same plane, be cut by a plane parallel to the common plane of their bases, the areas of the sections will be proportional to the areas of the bases; and if the bases are equivalent, the sections will also be equivalent.*

THEOREM XIII.

If two triangular pyramids have equivalent bases and equal altitudes, they are equal in volume.

Let V—ABC and v—abc be two triangular pyramids, having the equivalent bases, ABC and abc, and let the altitude of each be equal to CX; then will these two pyramids be equivalent.



Place the bases of the pyramids on the same plane, with their vertices in the same direction, and divide the altitude into any number of equal parts. Through the points of division pass planes parallel to the plane of the bases; the corresponding sections made in the pyramids by these planes are equivalent, (Th. 12, Cor. 2); that is, the triangle DEF is equivalent to the triangle def, the triangle GHI to the triangle ghi, etc.

Now, let triangular prisms be constructed on the triangles ABC, DEF, etc., of the pyramid $V—ABC$, these prisms having their lateral edges parallel to the edge, VC, of the pyramid, and the equal parts of the altitude, CX, for their altitudes. Portions of these prisms will be exterior to the pyramid $V—ABC$, and the sum of their volumes will exceed the volume of the pyramid.

On the bases def, ghi, etc., in the other pyramid, construct interior prisms, as represented in the figure, their lateral edges being parallel to vc, and their altitudes also the equal parts of the altitude, CX. Portions of the pyramid, $v—abc$, will be exterior to these prisms, and the volume of the pyramid will exceed the sum of the volumes of the prisms.

Since the sum of the exterior prisms, constructed in connection with the pyramid $V—ABC$, is greater than the pyramid, and the sum of the interior prisms, constructed in connection with the pyramid $v—abc$, is less than this pyramid, it follows that the difference of these sums is greater than the difference of the pyramids themselves. But the second exterior prism, or that on the base DEF, is equivalent to the first interior prism, or that on the base def, and the third exterior prism is equivalent to the second interior prism, (Th. 10, Cor. 2), and so on. That is, beginning with the second prism from the base of the pyramid, $V—ABC$, and taking these prisms in order towards the vertex of the pyramid, and comparing them with the prisms in the pyramid, $v—abc$, beginning with the lowest, and taking them in order toward the vertex of this pyramid, we find that to each exterior prism of the pyramid, $V—ABC$, exclusive of the first or lowest, there is a corresponding equivalent interior prism in the pyramid, $v—abc$.

Hence the prism, $ABCDEF$, is the difference between the sum of the prisms constructed in connection with the pyramid, $V—ABC$, and the sum of the interior prisms constructed in the pyramid, $v—abc$. But the first sum being a volume greater than the pyramid, $V—ABC$, and the second sum a volume less than the pyramid, $v—abc$, it follows that the volumes of the pyramids differ by less than the prism, $ABCDEF$.

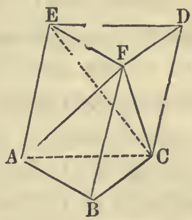
Now, however great the number of equal parts into which the altitude, CX, be divided, and the corresponding number of prisms constructed in connection with each pyramid, it would still be true that the difference between the volumes of the pyramids would be less than the volume of the lowest prism of the pyramid $V—ABC$; but when we make the number of equal parts into which the altitude is divided indefinitely great, the volume of this prism becomes indefinitely small: that is, the difference between the volumes of the pyramids is less than an indefinitely small volume; or, in other words, there is no assignable difference between the two pyramids, and they are, therefore, equivalent.

Hence the theorem; *if two triangular pyramids, etc.*

THEOREM XIV.

Any triangular pyramid is one third of the triangular prism having the same base and equal altitude.

Let $F—ABC$ be a triangular pyramid, and through F pass a plane parallel to the plane of the base, ABC. In this plane, through F, construct the triangle, FDE, having its sides, FD, DE, and EF, parallel and equal to BC, CA, and AB, respectively. The triangle, FDE, may be taken as the upper base of a triangular prism of which the lower base is ABC.



Now, this triangular prism is composed of the given triangular pyramid, $F—ABC$, and of the quadrangular pyramid, $F—ACDE$. This last pyramid may be divided by a plane through the three points, C, E, and F, into the two triangular pyramids, $F—DEC$ and $F—ACE$. But the pyramid, $F—DEC$, may be regarded as having the triangle, EFD, equal to the triangle, ABC, for its base, and the point, C, for its vertex. The two pyramids, $F—ABC$ and $C—DEF$, have equal bases and equal altitudes; they are therefore equivalent, (Th. 13). Again, the two pyramids, $F—DEC$ and $F—ACE$, have a common vertex, and equivalent bases in the same plane, and they are also equivalent. Therefore, the triangular prism, $ABCDEF$, is composed of three equivalent triangular pyramids, one of which is the given triangular pyramid, $F—ABC$.

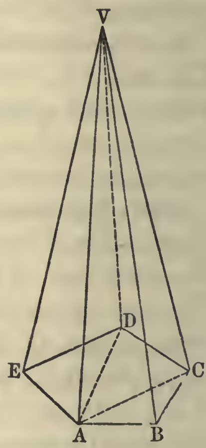
Hence the theorem; *any triangular pyramid is one third of the triangular prism, etc.*

Cor. The volume of the triangular prism being measured by the product of its base and altitude, *the volume of a triangular pyramid is measured by one third of the product of its base and altitude.*

THEOREM XV.

The volume of any pyramid whatever is measured by one third of the product of its base and altitude.

Let $V—ABCDE$ be any pyramid; then will its volume be measured by one third of the product of its base and altitude.



In the base of the pyramid, draw the diagonals, AD and AC, and through its vertex and these diagonals, pass planes, thus dividing the pyramid into a number of triangular pyramids having the common vertex V, and the altitude of the given pyramid for their common altitude.

Now, each of these triangular pyramids is measured by one third of the product of its base and altitude, (Cor., Th. 14), and their sum, which constitutes the polygonal pyramid, is therefore measured by one third of the product of the sum of the triangular bases and the common altitude; but the sum of the triangular bases constitutes the polygonal base, $ABCDE$.

Hence the theorem; *the volume of any pyramid whatever, etc.*

Cor. 1. Denote, by B, H, and V, respectively, the base, altitude, and volume of one pyramid, and by B', H', and

V', the base, altitude, and volume of another; then we shall have

$$V = \tfrac{1}{3}B \times H,$$

and $$V' = \tfrac{1}{3}B' \times H'.$$

Dividing the first of these equations by the second, member by member, we have

$$\frac{V}{V'} = \frac{B \times H}{B' \times H'},$$

which, in the form of a proportion, gives

$$V : V' :: B \times H : B' \times H'.$$

From this proportion we deduce the following consequences:

1st. *Pyramids are to each other as the products of their bases and altitudes.*

2d. *Pyramids having equivalent bases are to each other as their altitudes.*

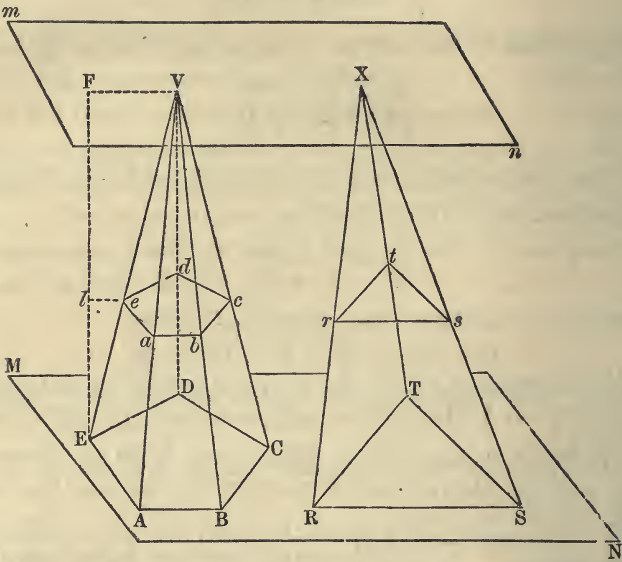
3d. *Pyramids having equal altitudes are to each other as their bases.*

Cor. 2. Since a prism is measured by the product of its base and altitude, and a pyramid by one third of the product of its base and altitude, we conclude that *any pyramid is one third of a prism having an equivalent base and equal altitude*

THEOREM XVI.

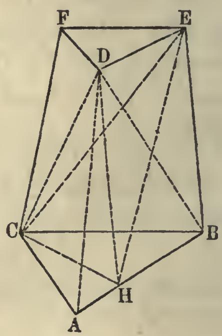
The volume of the frustum of a pyramid is equivalent to the sum of the volumes of three pyramids, each of which has an altitude equal to that of the frustum, and whose bases are, respectively, the lower base of the frustum, the upper base of the frustum, and a mean proportional between these bases.

Let $V—ABCDE$ and $X—RST$ be two pyramids, the one polygonal and the other triangular, having equivalent bases and equal altitudes; and let their bases be placed on the plane MN, their vertices falling on the parallel plane mn. Pass through the pyramids a plane



parallel to the common plane of their bases, cutting out the sections *abcde* and *rst*; these sections are equivalent, (Th. 12, Cor. 2), and the pyramids, *V—abcde* and *X—rst*, are equivalent, (Th. 13). Now, since the pyramids, *V—ABCDE* and *X—RST*, are equivalent, if from the first we take the pyramid, *V—abcde*, and from the second, the pyramid, *X—rst*, the remainders, or the frusta, *ABCDE—a* and *RST—r*, will be equivalent.

If, then, we prove the theorem in the case of the frustum of a triangular pyramid, it will be proved for the frustum of any pyramid whatever.



Let *ABC—D* be the frustum of a triangular pyramid. Through the points *D*, *B*, and *C*, pass a plane, and through the points *D*, *C*, and *E*, pass another, thus dividing the frustum into three triangular pyramids, viz., *D—ABC*, *C—DEF*, and *D--BEC*.

Now, the first of these has, for its

base, the lower base of the frustum, and for its altitude the altitude of the frustum, since its vertex is in the upper base; the second has, for its base, the upper base of the frustum, and for its altitude the altitude of the frustum, since its vertex is in the lower base. Hence, these are two of the three pyramids required by the enunciation of the theorem; and we have now only to prove that the third is equivalent to one having, for its base, a mean proportional between the bases of the frustum, and an altitude equal to that of the frustum.

In the face $ABED$, draw HD parallel to BE, and draw HE and HC. The two pyramids, $D—BEC$ and $H—BEC$, are equivalent, since they have a common base and equal altitudes, their vertices being in the line DH, which is parallel to the plane of their common base, (Th. 7, B. VI). We may, therefore, substitute the pyramid, $H—BEC$, for the pyramid, $D—BEC$. But the triangle, BCH, may be taken as the base, and E as the vertex of this new pyramid; hence, it has the required altitude, and we must now prove that it has the required base.

The triangles, ABC and HBC, have a common vertex, and their bases in the same line; hence, (Th. 16, B. II),

$$\triangle ABC : \triangle HBC :: AB : HB :: AB : DE. \quad (1)$$

In the triangles, DEF and HBC, $\angle E = \angle B$, and $DE = HB$; hence, if DEF be applied to HBC, $\angle E$ falling on $\angle B$, and the side DE on HB, the point D will fall on H, and the triangles, in this position, will have a common vertex, H, and their bases in the same line; hence,

$$\triangle HBC : \triangle DEF :: BC : EF. \quad (2)$$

But, because the triangles, ABC and DEF, are similar, we have

$$AB : DE :: BC : EF. \quad (3)$$

From proportions (1), (2), and (3), we have, (Th. 6, B. II),

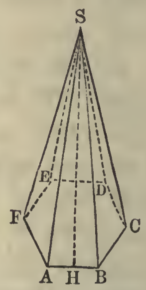
$$\triangle ABC : \triangle HBC :: \triangle HBC : \triangle DEF;$$

that is, the base, HBC, is a mean proportional between the lower and upper bases of the frustum.

Hence the theorem; *the volume of the frustum of a pyramid, etc.*

THEOREM XVII.

The convex surface of any right pyramid is measured by the perimeter of its base, multiplied by one half its slant height.



Let S—$ABCDEF$ be a right pyramid, of which SH is the slant height; then will its convex surface have, for its measure,

$$\tfrac{1}{2}SH(AB + BC + CD + DE + EF + FA).$$

Since the base is a regular polygon, and the perpendicular, drawn to its plane from S, passes through its center, the edges, SA, SB, SC, etc., are equal, (Th. 4, B. VI), and the triangles SAB, SBC, etc., are equal, and isosceles, each having an altitude equal to SH.

Now, $AB \times \frac{1}{2}SH$ measures the area of the triangle, SAB; and $BC \times \frac{1}{2}SH$ measures the area of the triangle, SBC; and so on, for the other triangular faces of the pyramid. By the addition of these different measures, we get

$$\tfrac{1}{2}SH(AB + BC + CD + DE + EF + FA),$$

as the measure of the total convex surface of the pyramid.

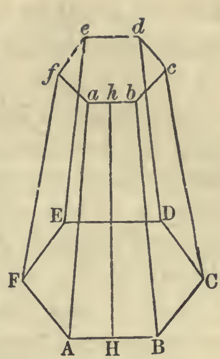
Hence the theorem; *the convex surface of any right pyramid, etc.*

THEOREM XVIII.

The convex surface of the frustum of any right pyramid is measured by the sum of the perimeters of the two bases, multiplied by one half the slant height of the frustum.

Let $ABCDEF$—d be the frustum of a right pyramid; then will its convex surface be measured by

$$\tfrac{1}{2}Hh(AB + BC + CD + DE + EF + FA + ab + bc + cd + de + ef + fa).$$



For, the upper base, *abcdef*, of the frustum is a section of a pyramid by a plane parallel to the lower base, (Def. 14), and is, therefore, similar to the lower base, (Th. 12). But the lower base is a regular polygon, (Def. 12); hence, the upper base is also a regular polygon, of the same name; and as ab and AB are intersections of a face of the pyramid by two parallel planes, they are parallel. For the same reason, bc is parallel to BC, cd to CD, etc., and the lateral faces of the frustum are all equal trapezoids, each having an altitude equal to Hh, the slant height of the frustum.

The trapezoid $ABba$ has, for its measure, $\frac{1}{2}Hh(AB+ab)$, (Th. 34, Book I); the trapezoid $BCcb$ has, for its measure, $\frac{1}{2}Hh(BC+bc)$, and so on, for the other lateral faces of the frustum.

Adding all these measures, we find, for their sum, which is the whole convex surface of the frustum,

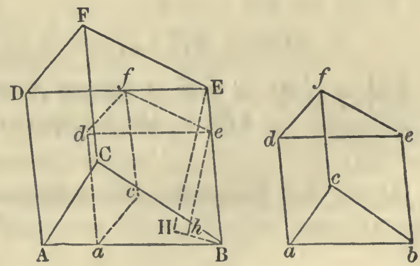
$$\tfrac{1}{2}Hh(AB+BC+CD+DE+EF+FA+ab+bc+cd+de+ef+fa).$$

Hence the theorem; *the convex surface of the frustum, etc.*

THEOREM XIX.

The volumes of similar triangular prisms are to each other as the cubes constructed on their homologous edges.

Let $ABC—F$ and $abc—f$ be two similar triangular prisms; then will their volumes be to each other as the cubes, whose edges are the homologous edges



AB and ab, or as the cubes, whose edges are the homologous edges BE and be, etc. Since the prisms are similar, the solid angles, whose vertices are B and b, are equal; and the smaller prism, when so applied to the larger that these solid angles coincide, will take, within the larger, the position represented by the dotted lines. In this position of the prisms, draw EH perpendicular to the plane of the base ABC, and join the foot of the perpendicular to the point B, and in the triangle BEH draw, through e, the line eh, parallel to EH; then will EH represent the altitude of the larger prism, and eh that of the smaller.

Now, as the bases ABC and aBc, are homologous faces, they are similar, and we have, (Th. 20, Book II),

$$\triangle ABC : \triangle aBc :: \overline{AB}^2 : \overline{aB}^2 \qquad (1)$$

But the $\triangle$'s BEH and Beh are equiangular, and therefore similar, and their homologous sides give the proportion

$$BE : Be :: EH : eh \qquad (2)$$

and from the homologous sides of the similar faces, $ABED$ and $aBed$, we also have

$$BE : Be :: AB : aB \qquad (3)$$

Proportions (2) and (3), having an antecedent and consequent the same in both, we have, (Th. 6, B. II),

$$EH : eh :: AB : aB \qquad (4)$$

By the multiplication of proportions (1) and (4), term by term, we get

$$\triangle ABC \times EH : \triangle aBc \times eh :: \overline{AB}^3 : \overline{aB}^3$$

But $\triangle ABC \times EH$ measures the volume of the larger prism, and $\triangle aBc \times eh$ measures the volume of the smaller.

Hence the theorem; *the volumes of similar triangular prisms, etc.*

Cor. 1. *The volumes of two similar prisms having any bases whatever, are to each other as the cubes constructed on their homologous edges.*

For, if planes be passed through any one of the lateral edges, and the several diagonal edges, of one of these prisms, this prism will be divided into a number of smaller triangular prisms. Taking the homologous edge of the other prism, and passing planes through it and the several diagonal edges, this prism will also be divided into the same number of smaller triangular prisms, similar to those of the first, each to each, and similarly placed.

Now, the similar smaller prisms, being triangular, are to each other as the cubes of their homologous edges; and being like parts of the larger prisms, it follows that the larger prisms are to each other as the cubes of the homologous edges of any two similar smaller prisms. But the homologous edges of the similar smaller prisms are to each other as the homologous edges of the given prisms; hence we conclude that the given prisms are to each other as the cubes of their homologous edges.

Cor. 2. *The volumes of two similar pyramids having any bases whatever, are to each other as the cubes constructed on their homologous edges.*

For, since the pyramids are similar, their bases are similar polygons; and upon them, as bases, two similar prisms may be constructed, having for their altitudes, the altitudes of their respective pyramids, and their lateral edges parallel to any two homologous lateral edges of the pyramids.

Now, these similar prisms are to each other as the cubes of their homologous edges, which may be taken as the homologous sides of their bases, or as their lateral edges, which were taken equal and parallel to any two arbitrarily assumed homologous lateral edges of the two pyramids; hence the pyramids which are thirds of their respective prisms, are to each other as the cubes constructed on any two homologous edges.

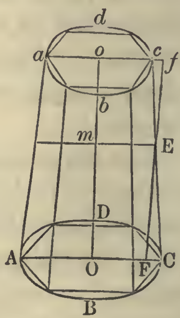
Cor. 3. *The volumes of any two similar polyedrons are to each other as the cubes constructed on their homologous edges.*

For, by passing planes through the vertices of the homologous solid angles of such polyedrons, they may both be divided into the same number of triangular pyramids, those of the one similar to those of the other, each to each, and similarly placed.

Now, any two of these similar triangular pyramids are to each other as the cubes of their homologous edges; and being like parts of their respective polyedrons, it follows that the polyedrons are to each other as the cubes of the homologous edges of any two of the similar triangular pyramids into which they may be divided. But the homologous edges of the similar triangular pyramids are to each other as the homologous edges of the polyedrons; hence the polyedrons are to each other as the cubes of their homologous edges.

THEOREM XX.

The convex surface of the frustum of a cone is measured by the product of the slant height and one half the sum of the circumferences of the bases of the frustum.



Let $ABCD$—$abcd$ be the frustum of a cone; then will its convex surface be measured by $Aa \times \frac{(\text{circ. } OC + \text{circ. } oc)}{2}$, in which the expression, circ. OC, denotes the circumference of the circle of which OC is the radius. Inscribe in the lower base of the frustum, a regular polygon having any number of sides, and in the upper base a similar polygon, having its sides parallel to those of the polygon in the lower base. These polygons may be taken as the bases of the frustum of a right pyramid inscribed in the frustum of the cone.

Now, however great the number of sides of the inscribed polygons, the convex surface of the frustum of the pyramid is measured by its slant height multiplied by one half the sum of the perimeters of its two bases, (Th. 18); but when we reach the limit, by making the number of sides of the polygon indefinitely great, the slant height, perimeters of the bases, and convex surface of the frustum of the pyramid become, severally, the slant height, circumferences of the bases, and convex surface of the frustum of the cone.

Hence the theorem; *the convex surface of the frustum, etc.*

Cor. 1. If we make $oc = OC$, and, consequently, circ. $oc =$ circ. OC, the frustum of the cone becomes a cylinder, and the half sum of the circumferences of the bases becomes the circumference of either base of the cylinder, and the slant height of the frustum, the altitude of the cylinder. Hence, *the convex surface of a cylinder is measured by the circumference of the base multiplied by the altitude of the cylinder.*

Cor. 2. If we make $oc = 0$, the frustum of the cone becomes a cone. Hence, *the convex surface of a cone is measured by the circumference of the base multiplied by one half the slant height of the cone.*

Cor. 3. If through E, the middle point of Cc, the line Ff be drawn parallel to Oo, and Em perpendicular to Oo, the line oc being produced, to meet Ff at f, we have, because the $\triangle$'s EFC and Efc are equal,

$$Em = \frac{OC + oc}{2}.$$

If we multiply both members of this equation by 2π, we have

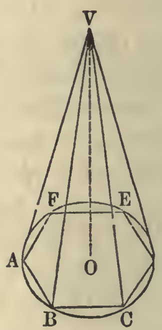
$$2\pi.Em = \frac{2\pi.OC + 2\pi.oc}{2};$$

that is, circ. *Em* is equal to one half the sum of the circumferences of the two bases of the frustum. Hence, *the convex surface of the frustum of a cone is measured by the circumference of the section made by a plane half way between the two bases, and parallel to them, multiplied by the slant height of the frustum.*

Cor. 4. If the trapezoid, *OCco*, be revolved about *Oo* as an axis, the inclined side, *Cc*, will generate the convex surface of the frustum of a cone, of which the slant height is *Cc*, and the circumferences of the bases are circ. *OC* and circ. *oc*. Hence, *if a trapezoid, one of whose sides is perpendicular to the two parallel sides, be revolved about the perpendicular side as an axis, it will generate the frustum of a cone, the inclined side opposite the axis generating the convex surface, and the parallel sides the bases of the frustum.*

THEOREM XXI.

The volume of a cone is measured by the area of its base multiplied by one third of its altitude.



Let *V—ABC*, etc., be a cone; then will its volume be measured by area *ABC*, etc., multiplied by $\frac{1}{3} VO$.

Inscribe, in the base of the cone, any regular polygon, as *ABCDEF*, which may be taken as the base of a right pyramid, of which *V* is the vertex. The volume of this inscribed pyramid will have, for its measure, (Th. 15),

$$\text{polygon } ABCDEF \times \tfrac{1}{3} VO.$$

Now, however great the number of sides of the polygon inscribed in the base of the cone, it will still hold true that the pyramid of which it is the base, and whose vertex is *V*, will be measured by the area of the polygon, multiplied by one third of *VO*; but when we reach the limit, by making the number of sides indefinitely great, the polygon becomes the circle in which it is inscribed, and the pyramid become the cone.

Hence the theorem; *the volume of a cone, etc.*

Cor. 1. If R denote the radius of the base of a cone, and H its altitude, or axis, its volume will be expressed by

$$\tfrac{1}{3}H \times \pi R^2;$$

hence, if V and V' designate the volumes of two cones, of which R and R' are the radii of the bases, and H and H' the altitudes, we have

$$V : V' :: \tfrac{1}{3}H \times \pi R^2 : \tfrac{1}{3}H' \times \pi R'^2 :: H \times \pi R^2 : H' \times \pi R'^2.$$

From this proportion we conclude,

First. *That cones having equal altitudes are to each other as their bases.*

Second. *That cones having equal bases are to each other as their altitudes.*

Cor. 2. Retaining the notation above, we have

$$\frac{V'}{V} = \frac{H'}{H} \times \frac{R'^2}{R^2}; \quad (1)$$

and, if the two cones are similar,

$$H : H' :: R : R';$$

or, $$\frac{H'}{H} = \frac{R'}{R}; \text{ hence, } \frac{H'^2}{H^2} = \frac{R'^2}{R^2}.$$

By substituting for the factors, in the second member of eq. (1), their values successively, and resolving into a proportion, we get

$$V : V' :: R^3 : R'^3;$$

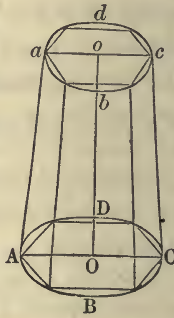
and $$V : V' :: H^3 : H'^3.$$

Hence, *similar cones are to each other as the cubes of the radii of their bases, and also as the cubes of their altitudes.*

Cor. 3. *A cone is equivalent to a pyramid having an equivalent base and an equal altitude.*

THEOREM XXII.

The volume of the frustum of a cone is equivalent to the sum of the volumes of three cones, having for their common altitude the altitude of the frustum, and for their several bases, the bases of the frustum and a mean proportional between them.



Let $ABCD$—$abcd$ be the frustum of a cone; then will its volume be equivalent to the sum of the volumes, having Oo for their common altitude, and for their bases, the circles of which, OC, oc, and a mean proportional between OC and oc, are the respective radii.

Inscribe in the lower base of the frustum any regular polygon, and in the upper base a similar polygon, having its sides parallel to those of the first. These polygons may be taken as the bases of the frustum of a right pyramid inscribed in the frustum of the cone.

The volume of the frustum of the pyramid is equivalent to the sum of the volumes of three pyramids, having for their common altitude the altitude of the frustum, and for their several bases the bases of the frustum, and a mean proportional between them, (Th. 16).

Now, however great the number of sides of the polygons inscribed in the bases of the frustum of the cone, this measure for the volume of the frustum of the pyramid, of which they are the bases, still holds true; but when we reach the limit, by making the number of the sides of the polygon indefinitely great, the polygons become the circles, the frustum of the pyramid becomes the frustum of the cone, and the three partial pyramids, whose sum is equivalent to the frustum of the pyramid, become three partial cones, whose sum is equivalent to the frustum of the cone.

Hence the theorem; *the volume of the frustum of a cone, etc.*

Cor. 1. Let R denote the radius of the lower base, R' that of the upper base, and H the altitude of the frustum of a cone; then will its volume be measured, (Th. 21), by

$$\tfrac{1}{3}H \times \pi R^2 + \tfrac{1}{3}H \times \pi R'^2 + \tfrac{1}{3}H \times \pi R \times R',$$

since $\pi R \times R'$ expresses the area of a circle which is a mean proportional between the two circles, whose radii are R and R'.

Now, if the bases of the frustum become equal, or $R = R'$, the frustum becomes a cylinder, and each of the last two terms in the above expression for the volume of the frustum of a cone will be equal to the first; hence, the volume of a cylinder, of which H is the altitude, and R the radius of the base, is measured by $H \times \pi R^2$.

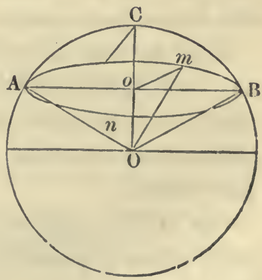
Therefore, *the volume of a cylinder is measured by the area of its base multiplied by its altitude.*

Cor. 2. By a process in all respects similar to that pursued in the case of cones, it may be shown that *similar cylinders are to each other as the cubes of the radii of their bases*, and also *as the cubes of their altitudes.*

Cor. 3. *A cylinder is equivalent to a prism having an equivalent base and an equal altitude.*

THEOREM XXIII.

If a plane be passed through a sphere, the section will be a circle.



Let O be the center of a sphere through which a plane is passed, making the section $AmBn$; then will this section be a circle.

From O let fall the perpendicular Oo upon the secant plane, and draw the radii OA, OB, and Om, to different points in the intersection of the plane with the surface of the sphere. Now, the oblique lines OA, OB, Om, are all equal, being radii of the sphere; they therefore meet the plane at equal dis tances from the foot of the perpendicular Oo, (Cor., Th. 4, B. VI); hence oA, oB, om, etc., are equal: that is, all the points in the intersection of the plane with the surface of the sphere are equally distant from the point O. This intersection is therefore the circumference of a circle of which o is the center.

Hence the theorem; *if a plane be passed through a sphere, etc.*

Cor. 1. Since AB, the diameter of the section, is a chord of the sphere, it is less than the diameter of the sphere; except when the plane of the section passes through the center of the sphere, and then its diameter becomes the diameter of the sphere. Hence,

1. *All great circles of a sphere are equal.*

2. *Of two small circles of a sphere, that is the greater whose plane is the less distant from the center of the sphere.*

3. *All the small circles of a sphere whose planes are at the same distance from the center, are equal.*

Cor. 2. Since the planes of all great circles of a sphere pass through its center, the intersection of two great circles will be both a diameter of the sphere and a common diameter of the two circles. Hence, *two great circles of a sphere bisect each other.*

Cor. 3. *A great circle divides the volume of a sphere, and also its surface, equally.*

For, the two parts into which a sphere is divided by any of its great circles, on being applied the one to the other, will exactly coincide, otherwise all the points in their convex surfaces would not be equally distant from the center.

Cor. 4. *The radius of the sphere which is perpendicular to the plane of a small circle, passes through the center of the circle.*

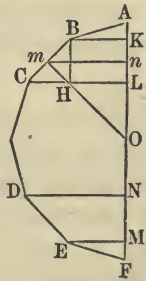
Cor. 5. *A plane passing through the extremity of a radius of a sphere, and perpendicular to it, is tangent to the sphere.*

For, if the plane intersect the sphere, the section is a circle, and all the lines drawn from the center of the sphere to points in the circumference are radii of the sphere, and are therefore equal to the radius which is perpendicular to the plane, which is impossible, (Cor. 1, Th. 3, B. VI). Hence the plane does not intersect the sphere, and has no point in its surface except the extremity of the perpendicular radius. The plane is therefore tangent to the sphere by Def. 22.

THEOREM XXIV.

If the line drawn through the center and vertices of two opposite angles of a regular polygon of an even number of sides, be taken as an axis of revolution, the perimeter of either semi-polygon thus formed will generate a surface whose measure is the axis multiplied by the circumference of the inscribed circle.

Let $ABCDEF$ be a semi-polygon cut off from a regular polygon of an even number of sides by drawing the line AF through the center O, and the vertices A and F, of two opposite angles of the polygon; then will the surface generated by the perimeter of this semi-polygon revolving about AF as an axis, be measured by $AF \times$ circumference of the inscribed circle.



From m, the middle point, and the extremities B and C of the side BC, draw mn, BK, and CL, perpendicular to AF; join also m and O, and draw BH perpendicular to CL. The surface of the frustum of the cone generated by the trapezoid $BKLC$, has for its measure circ. $mn \times BC$, (Cor. 3, Th. 20). Since mO is perpendicular to BC, and mn to BH, the two $\triangle$'s, BCH and mnO, are similar, and their homologous sides give the proportion

$$mn : mO :: BH\ (= KL) : BC$$

and as circumferences are to each other as their radii, we have

$$\text{circ. } mn : \text{circ. } mO :: KL : BC$$

Hence, $\quad \text{circ. } mn \times BC = \text{circ. } mO \times KL.$

But mO is the radius of the circle inscribed in tne polygon. Hence, the surface generated by BC during the revolution of the semi-polygon, is measured by the circumference of the inscribed circle multiplied by KL, the part of the axis included between the two perpendiculars let fall upon it from the extremities B and C. The surface generated by any other side of the semi-polygon will be measured, in like manner, by the circumference of the inscribed circle multiplied by the corresponding part of the axis.

By adding the measures of the surfaces generated by the several sides of the semi-polygon, we get

$$\text{Circ. } mO \times (AK + KL + LN + NM + MF)$$

for the measure of the whole surface.

Hence the theorem; *if the line drawn through the center, etc.*

Cor. It is evident that the surface generated by any portion, as CD and DE, of the perimeter, is measured by circ. $mO \times LM$.

THEOREM XXV.

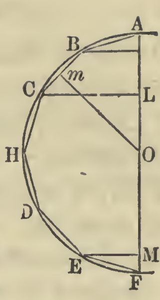
The surface of a sphere is measured by the circumference of one of its great circles multiplied by its diameter.

Let a sphere be generated by the revolution of the semi-circle, AHF, about its diameter, AF; then will the surface of the sphere be measured by

$$\text{Circ. } AO \times AF.$$

Inscribe in the semi-circle any regular semi-polygon, and let it be revolved, with the semi-circle, about the axis

AF; the surface generated by its perimeter will be measured by



$$\text{Circ. } mO \times AF, \text{ (Th. 24)},$$

and this measure will hold true, however great the number of sides of the inscribed semi-polygon. But as the number of these sides is increased, the radius mO, of the inscribed semi-circle, increases and approaches equality with the radius, AO; and when we reach the limit, by making the number of sides indefinitely great, the radii and semi-circles become equal, and the surface generated by the perimeter of the inscribed semi-polygon becomes the surface of the sphere. Therefore, the surface of the sphere has, for its measure,

$$\text{Circ. } AO \times AF.$$

Hence the theorem; *the surface of a sphere is measured, etc.*

Cor. 1. *A zone of a sphere is measured by the circumference of a great circle of the sphere multiplied by the altitude of the zone.*

For, the surface generated by any portion, as CD and DE, of the perimeter of the inscribed semi-polygon has, for its measure, circ. $mO \times LM$, (Cor. Th. 24); and as the number of the sides of the semi-polygon increases, LM remains the same, the radius mO alone changing, and becoming, when we reach the limit, equal to AO: hence, the surface of the zone is expressed by

$$\text{Circ. } AO \times LM,$$

whether the zone have two bases, or but one.

Cor. 2. Let H and H' denote the altitudes of two zones of spheres, whose radii are R and R'; then these zones will be expressed by $2\pi R \times H$ and $2\pi R' \times H'$; and if the surfaces of the zones be denoted by Z and Z', we have

$$Z : Z' : \ 2\pi R \times H : 2\pi R' \times H' :: R \times H : R' \times H'.$$

Hence, 1. *Zones in different spheres are to each other as their altitudes multiplied by the radii of the spheres.*

2. *Zones of equal altitudes are to each other as the radii of the spheres.*

3. *Zones in the same, or equal spheres, are to each other as their altitudes.*

Cor. 3. Let R denote the radius of a sphere; then will its diameter be expressed by $2R$, and the circumference of a great circle by $2\pi R$; hence its surface will be expressed by

$$2\pi R \times 2R = 4\pi R^2.$$

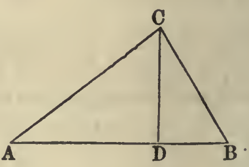
That is, *the surface of a sphere is equivalent to the area of four of its great circles.*

Cor. 4. *The surfaces of spheres are to each other as the squares of their radii.*

THEOREM XXVI.

If a triangle be revolved about either of its sides as an axis, the volume generated will be measured by one third of the product of the axis and the area of a circle, having for its radius the perpendicular let fall from the vertex of the opposite angle on the axis, or on the axis produced.

First. Let the triangle ABC, in which the perpendicular from C falls on the opposite side, AB, be revolved about AB as an axis; then will *Vol. $\triangle\ ABC$ have, for its measure, $\frac{1}{3}AB \times \pi\overline{CD}^2$.



The two $\triangle$'s into which $\triangle\ ABC$ is divided by the perpendicular DC, are right-angled, and during the revolution they will generate two cones, having for their

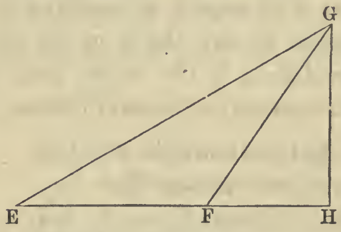
* Vol. $\triangle\ ABC$, cone $\triangle\ ADC$, are abbreviations for volume generated by $\triangle\ ABC$, cone generated by $\triangle\ ADC$; and surfaces of revolution generated by lines will hereafter be denoted by like abbreviations.

common base the circle, of which DC is the radius, and for their axes the parts DA and DB, into which AB is divided.

Now, *Cone $\triangle ADC$ is measured by $\frac{1}{3}AD \times \pi\overline{DC}^2$, (Th. 21), and cone $\triangle BDC$, by $\frac{1}{3}BD \times \pi\overline{DC}^2$; but these two cones compose Vol. $\triangle ABC$; and by adding their measures, we have, for that of Vol. $\triangle ABC$,

$$\tfrac{1}{3}AD \times \pi\overline{DC}^2 + \tfrac{1}{3}BD \times \pi\overline{DC}^2 = \tfrac{1}{3}AB \times \pi\overline{DC}^2.$$

Second. Let the triangle EFG, in which the perpendicular from G falls on the opposite side EF produced, be revolved about EF as an axis; then will Vol. $\triangle EFG$ have, for its measure, $\frac{1}{3}EF \times \pi\overline{GH}^2$, GH being the perpendicular on EF produced.



For, in this case it is apparent, that Vol. $\triangle EFG$ is the difference between the cone $\triangle EHG$ and the cone $\triangle FHG$. The first cone has, for its measure, $\frac{1}{3}EH \times \pi\overline{GH}^2$, and the second, for its measure, $\frac{1}{3}FH \times \pi\overline{GH}^2$; hence, by subtraction, we have

$$\text{Vol. } \triangle EFG = \tfrac{1}{3}EH \times \pi\overline{GH}^2 - \tfrac{1}{3}FH \times \pi\overline{GH}^2 = \tfrac{1}{3}EF \times \pi\overline{GH}^2.$$

Hence the theorem; *if a triangle be revolved about either of its sides, etc.*

Scholium.—If we take either of the above expressions for the measure of the volume generated by the revolution of a triangle about one of its sides, for example the last, and factor it otherwise, we have

$$\tfrac{1}{3}EF \times \pi\overline{GH}^2 = EF \times \tfrac{1}{2}GH \times \tfrac{1}{3}\pi \times 2GH = EF \times \tfrac{1}{2}GH \times \frac{2\pi \times GH}{3}.$$

Now, $EF \times \frac{1}{2}GH$ expresses the area of the triangle EFG; and $\frac{2\pi \times GH}{3}$, one third of the circumference described by the point G during the revolution.

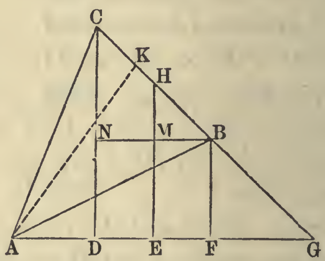
The expression, $\frac{1}{3}AB \times \pi\overline{DC}^2$, may be factored and interpreted in the

* See note on the preceding page.

same manner. Hence, we conclude that *the volume generated by the revolution of a triangle about either of its sides, is measured by the area of the triangle multiplied by one third of the circumference described in the revolution by the vertex of the angle opposite the axis.*

THEOREM XXVII.

The volume generated by the revolution of a triangle about any line lying in its plane, and passing through the vertex of one of its angles, is measured by the area of the triangle multiplied by two thirds of the circumference described, in the revolution, by the middle point of the side opposite the vertex through which the axis passes.



Let the triangle ABC be revolved about the line AG, drawn through the vertex A, and lying in the plane of the triangle, and let HE be the perpendicular let fall from H, the middle point of BC, upon the axis AG; then will Vol. $\triangle\ ABC$ have, for its measure, $\triangle\ ABC \times \frac{2}{3}$ circ. HE.

From the extremities of BC, let fall the perpendiculars BF and CD, on the axis; and from A draw AK perpendicular to BC, or BC produced, and produce CB, until it meets the axis in G.

Now, it is evident that Vol. $\triangle\ ABC$ is the difference between Vol. $\triangle\ AGC$ and Vol. $\triangle\ AGB$. But Vol. $\triangle\ AGC$ is expressed by $\triangle\ AGC \times \frac{1}{3}$ circ. CD; and Vol. $\triangle\ AGB$, by $\triangle\ AGB \times \frac{1}{3}$ circ. BF, (Scholium, Th. 26). Hence,

$$\text{Vol. } \triangle\ ABC = \triangle\ AGC \times \tfrac{1}{3} \text{ circ. } CD - \triangle\ AGB \times \tfrac{1}{3} \text{ circ. } BF.$$

Substituting for areas of $\triangle$'s, and for circumferences, their measures, we have

$$\text{Vol.}\ \triangle\ ABC = GC \times \tfrac{1}{2}AK \times \frac{2\pi.CD}{3} - GB \times \tfrac{1}{2}AK \times \frac{2\pi.BF}{3}$$

$$= GC \times \tfrac{1}{2}AK \times \frac{2\pi.CD}{3} - (GC - BC) \times \tfrac{1}{2}AK \times \frac{2\pi.BF}{3}$$

$$= GC \times \tfrac{1}{2}AK \times \frac{2\pi.CD}{3} - GC \times \tfrac{1}{2}AK \times \frac{2\pi.BF}{3} + BC \times \tfrac{1}{2}AK \times \frac{2\pi.BF}{3}$$

$$= GC \times \tfrac{1}{2}AK \times \frac{2\pi}{3}(CD - BF) + BC \times \tfrac{1}{2}AK \times \frac{2\pi.BF}{3}.$$

But BN being drawn parallel to AG, we have

$$CN = CD - BF;$$

hence, substituting this value for $CD - BF$, in the first term of the second member of the last equation, we have

$$\text{Vol.}\ \triangle\ ABC = GC \times \tfrac{1}{2}AK \times \frac{2\pi.CN}{3} + BC \times \tfrac{1}{2}AK \times \frac{2\pi.BF}{3}$$

$$= GC \times CN \times \tfrac{1}{2}AK \times \frac{2\pi}{3} + BC \times \tfrac{1}{2}AK \times \frac{2\pi.BF}{3},$$

by changing the order of factors in the first term of the second member. The homologous sides of the similar triangles, GCD and BCN, give the proportion

$$GC : CD :: BC : CN$$

whence, $\qquad GC \times CN = CD \times BC$

Substituting this value for $GC \times CN$, in the last equation above, and arranging the factors as before, it becomes

$$\text{Vol.}\ \triangle\ ABC = BC \times \tfrac{1}{2}AK \times \frac{2\pi.CD}{3} + BC \times \tfrac{1}{2}AK \times \frac{2\pi.BF}{3}$$

$$= BC \times \tfrac{1}{2}AK \times \frac{2\pi\,(CD + BF)}{3}.$$

But $CD + BF = 2HE$; hence

$$\text{Vol.}\ \triangle\ ABC = BC \times \tfrac{1}{2}AK \times \frac{4\pi.HE}{3} = BC \times \tfrac{1}{2}AK \times \tfrac{2}{3}.2\pi.HE;$$

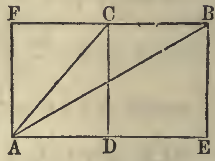
and since

$BC \times \frac{1}{2}AK = \triangle\ ABC$, and $\frac{2}{3} \times 2\pi.HE = \frac{2}{3}$ circ. HE,

this measure conforms to the enunciation.

It only remains for us to consider the case in which the axis is parallel to the base BC of the triangle The preceding demonstration will not now apply, because it supposes BC, or BC produced, to intersect the axis.

Let the axis AE, be parallel to the base BC, of the $\triangle$ ABC. From B and C let fall on the axis the perpendiculars BE and CD.



Now it is plain that

$$\text{Vol. } \triangle\ ABC = \text{cylinder rectangle } BCDE + \text{cone } \triangle\ ADC - \text{cone } \triangle\ AEB.$$

Substituting in second member, for cylinder and cones, their measures, we have

$$\text{Vol. } \triangle ABC = DE \times \pi\overline{CD}^2 + \tfrac{1}{3}AD \times \pi\overline{CD}^2 - \tfrac{1}{3}AE \times \pi\overline{BE}^2$$
$$= \tfrac{2}{3}DE \times \pi\overline{CD}^2 + \tfrac{1}{3}DE \times \pi\overline{CD}^2 + \tfrac{1}{3}AD \times \pi\overline{CD}^2 - \tfrac{1}{3}AE \times \pi\overline{BE}^2.$$

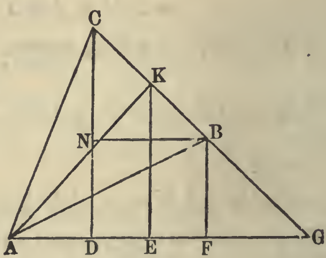
But $BE = CD$, and $\frac{1}{3}DE + \frac{1}{3}AD = \frac{1}{3}AE$. Reducing by these relations, we have

$$\text{Vol. } \triangle\ ABC = \tfrac{2}{3}DE \times \pi\overline{CD}^2 = \tfrac{1}{3}DE \times \tfrac{1}{2}CD \times 4\pi.CD$$
$$= DE \times \tfrac{1}{2}CD \times \tfrac{2}{3}.2\pi.CD = BC \times \tfrac{1}{2}CD \times \tfrac{2}{3}.2\pi.CD.$$

And, since $BC \times \frac{1}{2}CD$ expresses the area of the triangle ABC, and $\frac{2}{3}.2\pi.CD$, two thirds of the circumference described by any point of the base, this expression also conforms to the enunciation.

Hence the theorem; *the volume generated by the revolution, etc.*

Cor. If the generating triangle becomes isosceles, the perpendicular from A meets the base at its middle point. In this case, if we resume the expression



$$BC \times \tfrac{1}{2}AK \times \frac{4\pi.HE}{3},$$

it becomes

$$BC \times \tfrac{1}{2}AK \times KE \times \tfrac{4}{3}\pi.$$

But, since AK is perpendicular to BC, and KE to BN, the $\triangle$'s AKE and CBN are similar, and their homologous sides give the proportion

$$BC : BN :: AK : KE$$

whence, $$BC \times KE = BN \times AK$$

Changing the order of factors in the last expression on the preceding page, and replacing $BC \times KE$ by its value, it becomes

$$\tfrac{1}{2}AK \times AK \times BN \times \tfrac{4}{3}\pi = \overline{AK}^2 \times BN \times \tfrac{2}{3}\pi$$

Hence,

$$\text{Vol.} \triangle ABC = \tfrac{2}{3}\pi \times \overline{AK}^2 \times BN. = \tfrac{2}{3}\pi \times \overline{AK}^2 \times DF$$

That is, *the volume generated by the revolution of an isosceles triangle about any line drawn through its vertex and lying in the plane of the triangle, is measured by $\frac{2}{3}\pi$ times the square of the perpendicular of the triangle multiplied by the part of the axis included between the two perpendiculars let fall upon it from the extremities of the base of the triangle.*

Scholium.—If we resume the equation

$$\text{Vol.} \triangle ABC = BC \times \tfrac{1}{2}AK \times \frac{4\pi.HE}{3}$$

and change the order of the factors in the second member, it may be put under the form

$$\text{Vol.} \triangle ABC = BC \times 2\pi.HE \times \tfrac{1}{3}AK.$$

But during the revolution of the triangle, the side BC generates the surface of the frustum of a cone, which surface has for its measure

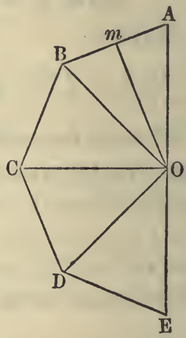
$$BC \times 2\pi.HE \text{ (Th. 20, Cor. 3).}$$

Hence, the above equation may be thus interpreted: *The volume generated by the revolution of a triangle about any line lying in its plane and passing through the vertex of one of its angles, is measured by the surface generated, during the revolution, by the side opposite the vertex through which the axis passes multiplied by one third of the perpendicular drawn from the vertex to that side.*

THEOREM XXVIII.

If the line drawn through the center and vertices of two opposite angles of a regular polygon, of an even number of sides, be taken as an axis of revolution, either semi-polygon thus formed will, during this revolution, generate a volume which has, for its measure, the surface generated by the perimeter of the semi-polygon multiplied by one third of its apothem.

Let $ABCDE$ be a regular semi-polygon, cut off from a regular polygon of an even number of sides, by drawing a line through the center, O, and the vertices, A and E, of two opposite angles of the polygon; then will the volume generated by the revolution of this semi-polygon about AE, as an axis, be measured by (Sur. AB + sur. BC + sur. CD + sur. DE) $\times \frac{1}{3} Om$, Om being the apothem of the polygon.



For, if from the center of O, the lines OB, OC, OD, be drawn to the vertices of the several angles of the semi-polygon, it will be divided into equal isosceles triangles, the perpendicular of each being the apothem of the polygon.

Now, the volume generated by $\triangle\, AOB$ has, for its measure,

Sur. $AB \times \frac{1}{3} Om$,

that by $\triangle\, BOC$, Sur. $BC \times \frac{1}{3} Om$,

" $\triangle\, COD$, Sur. $CD \times \frac{1}{3} Om$,

" $\triangle\, DOE$, Sur. $DE \times \frac{1}{3} Om$, (Scholium, Th. 27).

By the addition of the measures of these partial volumes, we find, for that of the whole volume,

Vol. semi-polygon $ABCDE$ = sur. perimeter $ABCDE \times \frac{1}{3} Om$,

and were the number of the sides of the semi-polygon increased or diminished, the reasoning would be in no wise changed.

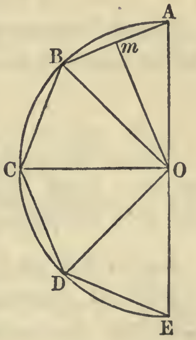
Hence the theorem; *if the line drawn through the center, etc.*

SCHOLIUM.—The volume generated by any portion of the semi-polygon, as that composed of the two isosceles $\triangle$'s BOC, COD, is measured by

$$\text{Sur. perimeter } BCD \times \tfrac{1}{3} Om.$$

THEOREM XXIX.

The volume of a sphere is measured by its surface multiplied by one third of its radius.



Let a sphere be generated by the revolution of the semicircle ACE, about its diameter, AE, as an axis; then will the volume of the sphere be measured by

$$\text{sur. semi-circ. } OA \times \tfrac{1}{3} OA.$$

For, inscribe in the semi-circle any regular semi-polygon, as $ABCDE$, and let it, together with the semi-circle, revolve about the axis AE. The semi-polygon will generate a volume which has, for its measure,

$$\text{Sur. perimeter } ABCDE \times \tfrac{1}{3} Om, \text{ (Th. 28)},$$

in which Om is the apothem of the polygon.

Now, however great the number of sides of the inscribed regular semi-polygon, this measure for the volume generated by it, will hold true; but when we reach the limit, by making the number of sides indefinitely great, the perimeter and apothem become, respectively, the semi-circumference and its radius, and the volume generated by the semi-polygon becomes that generated by the semi-circle, that is, the sphere. Therefore,

$$\text{Vol. sphere} = \text{sur. semi-circ. } OA \times \tfrac{1}{3} OA.$$

SCHOLIUM 1.—If we take any portion of the inscribed semi-polygon, as BOC, the volume generated by it is measured by sur. $BC \times \frac{1}{3} Om$, (Scholium, Th. 27); and when we pass to the limit, this volume becomes a sector, and sur. BC a zone of the sphere, which zone is the base of the sector. Hence, *the volume of a spherical sector is measured by the zone which forms its base multiplied by one third of the radius of the sphere.*

SCHOLIUM 2.—Let R denote the radius of a sphere; then will its diameter be represented by $2R$. Now, since the surface of a sphere is equivalent to the area of four of its great circles, and the area of a great circle is expressed by πR^2, we have

$$\text{Vol. sphere} = 4\pi R^2 \times \tfrac{1}{3}R = \tfrac{4}{3}\pi R^3.$$

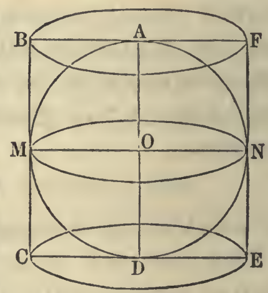
And since $R^3 = \frac{1}{8}(2R)^3$, we also have

$$\text{Vol. sphere} = \tfrac{4}{3}\pi R^3 = \tfrac{1}{6}\pi(2R)^3.$$

Hence, *the volume of a sphere is measured by four thirds of π times the cube of the radius, or by one sixth of π times the cube of the diameter.*

THEOREM XXX.

The surface of a sphere is equivalent to two thirds of the surface, bases included, and the volume of a sphere to two thirds of the volume, of the circumscribing cylinder.



Let AMD be a semi-circle, and $ABCD$ a rectangle formed by drawing tangents through the middle point and extremities of the semi-circumference, and let the semi-circle and rectangle be revolved together about AD as an axis. The rectangle will thus generate a cylinder circumscribed about the sphere generated by the semi-circle.

First. The diameter of the base, and the altitude of the cylinder, are each equal to the diameter of the sphere; hence the convex surface of the cylinder, being measured by the circumference of its base multiplied by its altitude, (Cor. 1, Th. 20), has the same measure as the surface of the sphere, (Th. 25). But the surface of the sphere is equivalent to four great circles, (Cor. 3, Th. 25). Hence, the convex surface of the cylinder is equivalent to four great circles; and adding to these the bases of the cylinder, also great circles, we have the whole surface of the cylinder equivalent to six great circles. Therefore, the surface of the sphere is four sixths = two thirds of the surface of the cylinder, including its bases.

Second. The volume of the cylinder, being measured by the area of the base multiplied by the altitude, (Cor. 1, Th. 22), is, in this case, measured by the area of a great circle multiplied by its diameter = four great circles multiplied by one half the radius of the sphere.

But the volume of the sphere is measured by four great circles multiplied by one third of the radius, (Scholium 2, Th. 29). Therefore,

$$\text{Vol. sphere} : \text{Vol. cylinder} :: \tfrac{1}{3} : \tfrac{1}{2} :: 2 : 3;$$

whence, $\text{Vol. sphere} = \tfrac{2}{3}\ \text{Vol. cylinder}.$

Hence the theorem; *the surface of a sphere is equivalent, etc.*

Cor. The volume of a sphere is to the volume of the circumscribed cylinder, as the surface of the sphere is to the surface of the cylinder.

Scholium.—Any polyedron circumscribing a sphere, may be regarded as composed of as many pyramids as the polyedron has faces, the center of the sphere being the common vertex of these pyramids, and the several faces of the polyedron their bases. The altitude of each pyramid will be a radius of the sphere; hence the volume of any one pyramid will be measured by the area of the face of the polyedron which forms its base, multiplied by one third of the radius of the sphere. Therefore, the aggregate of these pyramids, or the whole polyedron, will be measured by the surface of the polyedron multiplied by one third of the radius of the sphere.

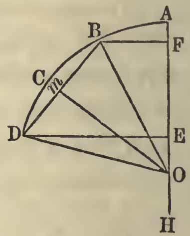
But the volume of the sphere is also measured by the surface of the sphere multiplied by one third of its radius. Hence,

Sur. polyedron : Sur. sphere :: Vol. polyedron : Vol. sphere.

That is, *the surface of any circumscribed polyedron is to the surface of the sphere, as the volume of the polyedron is to the volume of the sphere.*

THEOREM XXXI.

The volume generated by the revolution of the segment of a circle about a diameter of the circle exterior to the segment, is measured by one sixth of π times the square of the chord of the segment, multiplied by the part of the axis included between the perpendiculars let fall upon it from the extremities of the chord.



Let BCD be a segment of the circle, whose center is O, and AH a part of a diameter exterior to the segment. Draw the chord BD, and from its extremities let fall the perpendiculars, BF, DE on AH; also draw Om perpendicular to BD. The spherical sector generated by the revolution of the circular sector $BCDO$ about AH, is measured by zone $BD \times \frac{1}{3}BO$, (Scholium 1, Th. 29), $= 2\pi . BO \times EF \times \frac{1}{3}BO = \frac{2}{3}\pi\overline{BO}^2 \times EF$; and the volume generated by the isosceles triangle BOD is measured by

$$\tfrac{2}{3}\pi\overline{Om}^2 \times EF, \text{ (Cor. 1, Th. 27).}$$

The difference between these two volumes is that generated by the circular segment BCD, which has, therefore, for its measure,

$$\tfrac{2}{3}\pi EF(\overline{BO}^2 - \overline{Om}^2) = \tfrac{2}{3}\pi EF \times \overline{Bm}^2, \text{ (Th. 39, B. I).}$$

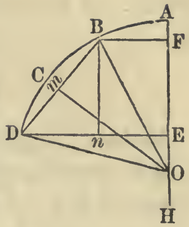
But since $Bm = \frac{1}{2}BD$, $\overline{Bm}^2 = \frac{1}{4}\overline{BD}^2$; hence, by substituting, we have

$$\text{Vol. segment } BCD = \tfrac{2}{3}\pi EF \times \tfrac{1}{4}\overline{BD}^2 = \tfrac{1}{6}\pi\overline{BD}^2 \times EF.$$

Hence the theorem.

THEOREM XXXII.

The volume of a segment of a sphere has, for its measure, the half sum of the bases of the segment multiplied by its altitude, plus the volume of a sphere which has this altitude for its diameter.



Let BCD be the arc of a circle, and BF and DE perpendiculars let fall from its extremities upon a diameter, of which AH is a part; then, if the area $BCDEF$ be revolved about AH as an axis, a spherical segment will be generated, for the volume of which it is proposed to find a measure.

The circular segment will generate a volume measured by $\frac{1}{6}\pi\overline{BD}^2 \times EF$, (Th. 31); and the frustum of the cone generated by the trapezoid $BDEF$ will have, for its measure,

$$\tfrac{1}{3}\pi\overline{BF}^2 \times EF + \tfrac{1}{3}\pi\overline{DE}^2 \times EF + \tfrac{1}{3}\pi BF \times DE \times EF, \text{(Th. 22)},$$
$$= \tfrac{1}{3}\pi EF(\overline{BF}^2 + \overline{DE}^2 + BF \times DE).$$

But the sum of these two volumes is the volume of the spherical segment, which has, therefore, for its measure,

$$\tfrac{1}{6}\pi EF\,(\overline{BD}^2 + 2\overline{BF}^2 + 2\overline{DE}^2 + 2BF \times DE)$$

From B let fall the perpendicular Bn on DE; then will

$$Dn = DE - nE = DE - BF;$$

hence, $\quad \overline{Dn}^2 = \overline{DE}^2 - 2DE \times BF + \overline{BF}^2;$

and since $\quad \overline{BD}^2 = \overline{Bn}^2 + \overline{Dn}^2 = \overline{EF}^2 + \overline{Dn}^2,$

we have $\quad \overline{BD}^2 = \overline{EF}^2 + \overline{DE}^2 + \overline{BF}^2 - 2DE \times BF.$

By substituting this value for $\overline{BD}^2$, in the above measure for the volume of the segment, we find

$$\tfrac{1}{6}\pi EF(\overline{EF}^2 + \overline{DE}^2 + \overline{BF}^2 - 2DE \times BF + 2\overline{BF}^2 + 2\overline{DE}^2 + 2BF \times DE)$$
$$= \tfrac{1}{6}\pi EF\,(\overline{EF}^2 + 3\overline{DE}^2 + 3\overline{BF}^2) = \tfrac{1}{6}\pi\overline{EF}^3 + EF\left(\frac{\pi\overline{DE}^2 + \pi\overline{BF}^2}{2}\right).$$

Which last expression conforms to the enunciation.

Hence the theorem; *the volume of a segment of a sphere, etc.*

Cor. When the segment has but one base, BF becomes zero, and EF becomes EA; and the final expression which we found for the volume of the segment reduces to

$$\tfrac{1}{6}\pi\overline{EA}^3 + EA \times \frac{\pi\overline{DE}^2}{2}.$$

Hence, *A spherical segment having but one base, is equivalent to a sphere whose diameter is the altitude of the segment, plus one half of a cylinder having for base and altitude the base and altitude of the segment.*

Scholium.—When the spherical segment has a single base, we may put the expression, $\tfrac{1}{6}\pi\overline{EA}^3 + EA \times \frac{\pi\overline{DE}^2}{2}$, under a form to indicate a convenient practical rule for computing the volume of the segment.

Thus, since the triangle DEO is right-angled, and $OE = OA - EA$, we have

$$\overline{DE}^2 = \overline{DO}^2 - \overline{OE}^2 = \overline{OA}^2 - \overline{OA}^2 + 2OA \times EA - \overline{EA}^2$$
$$= 2OA \times EA - \overline{EA}^2.$$

By substituting this value for $\overline{DE}^2$ in the expression for the volume of the segment, we find

$$\tfrac{1}{6}\pi\overline{EA}^3 + EA \times \frac{\pi}{2} \times (2OA \times EA - \overline{EA}^2)$$
$$= \tfrac{1}{6}\pi\overline{EA}^3 + \overline{EA}^2 \times \frac{\pi}{2}(2OA - EA)$$
$$= \tfrac{1}{6}\pi\overline{EA}^3 + \tfrac{1}{6}\pi.3\overline{EA}^2(2OA - EA)$$
$$= \tfrac{1}{6}\pi\overline{EA}^2(EA + 6.OA - 3EA)$$
$$= \tfrac{1}{6}\pi\overline{EA}^2(6.OA - 2EA)$$
$$= \tfrac{1}{3}\pi\overline{EA}^2(3OA - EA)$$

Hence, *the volume of a spherical segment, having a single base, is measured by one third of π times the square of the altitude of the segment, multiplied by the difference between three times the radius of the sphere and this altitude.*

RECAPITULATION

Of some of the principles demonstrated in this and the preceding Books.

Let R denote the radius, and D the diameter of any circle or sphere, and H the altitude of a cone, or of a segment of a sphere; then,

Circumference of a circle	$= 2\pi R.$
Surface of a sphere	$= 4\pi R^2$, or πD^2.
Zone forming the base of a segment of a sphere,	$= 2\pi R \times H.$
Volume or solidity of a sphere	$= \frac{4}{3}\pi R^3$, or $\frac{1}{6}\pi D^3$.
Volume of a spherical sector	$= \frac{2}{3}\pi R^2 \times H.$
Volume of a cone, of which R is the radius of the base	$= \frac{1}{3}\pi R^2 \times H.$
Volume of a spherical segment, of which R' is the radius of one base, and R'' the radius of the other, and whose altitude is H,	$= \frac{1}{6}\pi H^3 + H\frac{(\pi R'^2 + \pi R''^2)}{2}$
If the segment has but one base, $R'' =$ zero, and the volume of the segment,	$= \frac{1}{6}\pi H^3 + H.\frac{\pi R'^2}{2}$; or, $= \frac{1}{3}\pi H^2(3R - H)$.

PRACTICAL PROBLEMS.

1. The diameter of a sphere is 12 inches; how many cubic inches does it contain? *Ans.* 904.78 cu. in.

2. What is the solidity of the segment of a single base that is cut from a sphere 12 inches in diameter, the altitude of the segment being 3 inches? *Ans.* 141.372 cu. in.

3. The surface of a sphere is 68 square feet; what is its diameter? *Ans.* $D = 4.652$ feet.

4. If from a sphere, whose surface is 68 square feet, a segment be cut, having a depth of two feet and a single base, what is the convex surface of the segment?

Ans. 29.229+ sq. ft.

5. What is the solidity of the sphere mentioned in the two preceding examples, and what is the solidity of the segment, having a depth of two feet, and but one base?

Ans. { Solidity of sphere, 52.71 cu. ft.
" " segment, 20.85 "

6. In a sphere whose diameter is 20 feet, what is the solidity of a segment, the bases of which are on the same side of the center, the first at the distance of 3 feet from it, and the second of 5 feet; and what is the solidity of a second segment of the same sphere, whose bases are also on the same side of the center, and at distances from it, the first of 5 and the second of 7 feet?

Ans. { Solidity of first segment, 525.7 cu. ft.
" " second " 400.03 "

7. If the diameter of the single base of a spherical segment be 16 inches, and the altitude of the segment 4 inches, what is its solidity? *

Ans. 435.6352 cubic inches.

8. The diameter of one base of a spherical segment is 18 inches, and that of the other base 14 inches, these bases being on opposite sides of the center of the sphere, and the distance between them 9 inches; what is the volume of the segment, and the radius of the sphere?

Ans. { Vol. seg., 2219.5 cubic inches.
Rad. of sphere, 9.4027 inches.

9. The radius of a sphere is 20, the distance from the center to the greater base of a segment is 10, and the distance from the same point to the lesser base is 16; what is the volume of the segment, the bases being on the same side of the center? *Ans.* 4297.7088.

10. If the diameter of one base of a spherical segment be 20 miles, and the diameter of the other base 12 miles, and the altitude of the segment 2 miles, what is its solidity, and what is the diameter of the sphere?

---

* **First find the radius of the sphere.**

NOTE.—The KEY to this work contains full solutions to all the problems in the Geometry and Trigonometry, and the necessary diagrams for illustration.

BOOK VIII.

PRACTICAL GEOMETRY.

APPLICATION OF ALGEBRA TO GEOMETRY, AND ALSO PROPOSITIONS FOR ORIGINAL INVESTIGATION.

No definite rules can be given for the algebraic solution of geometrical problems. The student must, in a a great measure, depend on his own natural tact, and his power of making a skillful application of the geometrical and analytical knowledge he has thus far obtained.

The known quantities of the problem should be represented by the first letters of the alphabet, and the unknown by the final letters; and the relations between these quantities must be expressed by as many independent equations as there are unknown quantities. To obtain the equations of the problem, we draw a figure, the parts of which represent the known and unknown magnitudes, and very frequently it will be found necessary to draw auxiliary lines, by means of which we can deduce, from the conditions enunciated, others that can be more conveniently expressed by equations. In many cases the principal difficulty consists in finding, from the relations directly given in the statement, those which are ultimately expressed by the equations of the problem. Having found these equations, they are treated by the known rules of algebra, and the values of the required magnitudes determined in terms of those given.

PROBLEM I.

Given, the hypotenuse, and the sum of the other two sides of a right-angled triangle, to determine the triangle.

Let ABC be the $\triangle$. Put $CB = y$, $AB = x$, $AC = h$, and $CB + AB = s$. Then, by a given condition, we have

$$x + y = s;$$

and, $$x^2 + y^2 = h^2, \text{ (Th. 39, B. I).}$$

Reducing these two equations, and we have

$$x = \tfrac{1}{2}s \pm \tfrac{1}{2}\sqrt{2h^2 - s^2}; \qquad y = \tfrac{1}{2}s \pm \tfrac{1}{2}\sqrt{2h^2 - s^2}.$$

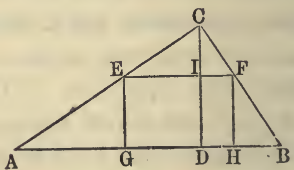
If $h = 5$ and $s = 7$, $x = 4$ or 3, and $y = 3$ or 4.

Remark.—In place of putting x to represent one side, and y the other, we might put $(x + y)$ to represent the greater side, and $(x - y)$ the less side; then,

$$x^2 + y^2 = \frac{h^2}{2}, \text{ and } 2x = s, \text{ etc.}$$

PROBLEM II.

Given, the base and perpendicular of a triangle, to find the side of its inscribed square.



Let ABC be the $\triangle$. Put $AB = b$, the base, $CD = p$, the perpendicular.

Draw EF parallel to AB, and suppose it equal to EG, a side of the required square; and put $EF = x$.

Then, by similar $\triangle$'s, we have

$$CI : EF :: CD : AB.$$

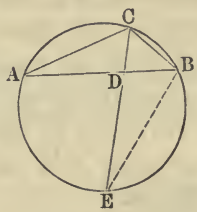
That is, $$p - x : x :: p : b.$$

Hence, $$bp - bx = px; \text{ or, } x = \frac{bp}{b + p}.$$

That is, *the side of the inscribed square is equal to the product of the base and altitude, divided by their sum.*

PROBLEM III.

In a triangle, having given the sides about the vertical angle, and the line bisecting that angle and terminating in the base, to find the base.



Let ABC be the $\triangle$, and let a circle be circumscribed about it. Divide the arc AEB into two equal parts at the point E, and draw EC. This line bisects the vertical angle, (Cor., Th. 9, B. III). Draw BE.

Put $AD = x$, $DB = y$, $AC = a$, $CB = b$, $CD = c$, and $DE = w$. The two $\triangle$'s, ADC and EBC, are equiangular; from which we have

$$w + c : b :: a : c; \text{ or, } cw + c^2 = ab; \quad (1)$$

But, as EC and AB are two chords that intersect each other in a circle, we have

$$cw = xy, \quad \text{(Th. 17, B. III).}$$

Therefore, $$xy + c^2 = ab. \quad (2)$$

But, as CD bisects the vertical angle, we have

$$a : b :: x : y, \quad \text{(Th. 24, B. II).}$$

Or, $$x = \frac{ay}{b}. \quad (3)$$

Hence, $$\frac{a}{b}y^2 + c^2 = ab; \text{ or, } y = \sqrt{b^2 - \frac{c^2b}{a}}$$

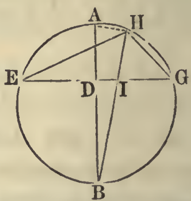
And, $$x = \frac{a}{b}\sqrt{b^2 - \frac{c^2b}{a}}.$$

Now, as x and y are determined, the base is determined.

REMARK.—Observe that equation (2) is Theorem 20, Book III

PROBLEM IV.

To determine a triangle, from the base, the line bisecting the vertical angle, and the diameter of the circumscribing circle.



Describe the circle on the given diameter, AB, and divide it into two parts, in the point D, so that $AD \times DB$ shall be equal to the square of one half the given base, (Th. 17, B. III).

Through D draw EDG, at right angles to AB, and EG will be the given base of the triangle.

Put $\qquad AD = n,\ DB = m,\ AB = d,\ DG = b.$

Then, $\qquad n + m = d$, and $nm = b^2$;

and these two equations will determine n and m; therefore, we shall consider n and m as known.

Now, suppose EHG to be the required △; and draw HIB and HA. The two △'s, ABH, DBI, are equiangular; and, therefore, we have

$$AB : HB :: IB : DB.$$

But HI is a given line, that we will represent by c; and if we put $IB = w$, we shall have $HB = c + w$; then the above proportion becomes,

$$d : c + w :: w : m.$$

Now, w can be determined by a quadratic equation; and, therefore, IB is a known line.

In the right-angled △ DBI, the hypotenuse IB, and the base DB, are known; therefore, DI is known, (Th. 39, B. I); and if DI is known, EI and IG are known.

Lastly, let $EH = x$, $HG = y$, and put $EI = p$, and $IG = q$.

Then, by Theorem 20, Book III, $pq + c^2 = xy \qquad (1)$

But, $\qquad x : y :: p : q$ (Th. 24, B. II)

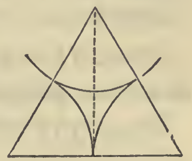
Or, $$x = \frac{py}{q} \qquad (2)$$

Now, from equations (1) and (2) we can determine x and y, the sides of the △; and thus the determination has been attained, carefully and easily, step by step.

PROBLEM V.

Three equal circles touch each other externally, and thus inclose one acre of ground; what is the diameter in rods of each of these circles?

Draw three equal circles to touch each other externally, and join the three centers, thus forming a triangle. The lines joining the centers will pass through the points of contact, (Th. 7, B. III).



Let R represent the radius of these equal circles; then it is obvious that each side of this △ is equal to $2R$. The triangle is therefore equilateral, and it incloses the given area, and three equal sectors.

As the angle of each sector is one third of two right angles, the three sectors are, together, equal to a semi-circle; but the area of a semi-circle, whose radius is R, is expressed by $\frac{\pi R^2}{2}$; and the area of the whole triangle must be $\frac{\pi R^2}{2} + 160$; but the area of the △ is also equal to R multiplied by the perpendicular altitude, which is $R\sqrt{3}$.

Therefore, $$R^2\sqrt{3} = \frac{\pi R^2}{2} + 160.$$

Or, $$R^2(2\sqrt{3} - \pi) = 320.$$

$$R^2 = \frac{320}{2\sqrt{3} - 3.1415926} = \frac{320}{0.3225} = 992.248.$$

Hence, $R = 31.48 +$ rods, for the required result.

PROBLEM VI. — *In a right-angled triangle, having given the base and the sum of the perpendicular and hypotenuse, to find these two sides.*

PROB. VII.—*Given, the base and altitude of a triangle, to divide it into three equal parts, by lines parallel to the base.*

PROB. VIII.—*In any equilateral* $\triangle$, *given the length of the three perpendiculars drawn from any point within, to the three sides, to determine the sides.*

PROB. IX.—*In a right-angled triangle, having given the base,* (3), *and the difference between the hypotenuse and perpendicular,* (1), *to find both these two sides.*

PROB. X. —*In a right-angled triangle, having given the hypotenuse,* (5), *and the difference between the base and perpendicular,* (1), *to determine both these two sides.*

PROB. XI.—*Having given the area of a rectangle inscribed in a given triangle, to determine the sides of the rectangle.*

PROB. XII.—*In a triangle, having given the ratio of the two sides, together with both the segments of the base, made by a perpendicular from the vertical angle, to determine the sides of the triangle.*

PROB. XIII.—*In a triangle, having given the base, the sum of the other two sides, and the length of a line drawn from the vertical angle to the middle of the base, to find the sides of the triangle.*

PROB. XIV.—*To determine a right-angled triangle, having given the lengths of two lines drawn from the acute angles to the middle of the opposite sides.*

PROB. XV.—*To determine a right-angled triangle, having given the perimeter, and the radius of the inscribed circle.*

PROB. XVI.—*To determine a triangle, having given the base, the perpendicular, and the ratio of the two sides.*

PROB. XVII.—*To determine a right-angled triangle, having given the hypotenuse, and the side of the inscribed square.*

PROB. XVIII. — *To determine the radii of three equal circles inscribed in a given circle, and tangent to each other, and also to the circumference of the given circle.*

PROB. XIX. —*In a right-angled triangle, having given the perimeter, or sum of all the sides, and the perpendicular let fall from the right angle on the hypotenuse, to determine the triangle; that is, its sides.*

PROB. XX.—*To determine a right-angled triangle, having given the hypotenuse, and the difference of two lines drawn from the two acute angles to the center of the inscribed circle.*

PROB. XXI. — *To determine a triangle, having given the base, the perpendicular, and the difference of the two other sides.*

PROB. XXII. — *To determine a triangle, having given the base, the perpendicular, and the rectangle, or product of the two sides.*

PROB. XXIII.—*To determine a triangle, having given the lengths of three lines drawn from the three angles to the middle of the opposite sides.*

PROB. XXIV. — *In a triangle, having given all the three sides, to find the radius of the inscribed circle.*

PROB. XXV.—*To determine a right-angled triangle, having given the side of the inscribed square, and the radius of the inscribed circle.*

PROB. XXVI. — *To determine a triangle, and the radius of the inscribed circle, having given the lengths of three lines drawn from the three angles to the center of that circle.*

PROB. XXVII. — *To determine a right-angled triangle, having given the hypotenuse, and the radius of the inscribed circle.*

PROB. XXVIII.—*The lengths of two parallel chords on the same side of the center being given, and their distance apart, to determine the radius of the circle.*

PROB. XXIX. — *The lengths of two chords in the same* *circle being given, and also the difference of their distances from the center, to find the radius of the circle.*

PROB. XXX.—*The radius of a circle being given, and also the rectangle of the segments of a chord, to determine the distance of the point at which the chord is divided, from the center.*

PROB. XXXI.—*If each of the two equal sides of an isosceles triangle be represented by* a, *and the base by* 2b, *what will be the value of the radius of the inscribed circle?*

$$\text{Ans. } R = \frac{b\sqrt{a^2 - b^2}}{a + b}.$$

PROB. XXXII. — *From a point without a circle whose diameter is* d, *a line equal to* d *is drawn, terminating in the concave arc, and this line is bisected at the first point in which it meets the circumference. What is the distance of the point without from the center of the circle?*

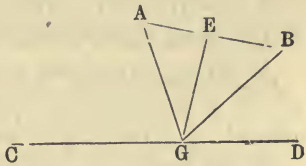
It is not deemed necessary to multiply problems in the application of algebra to geometry. The preceding will be a sufficient exercise to give the student a clear conception of the nature of such problems, and will serve as a guide for the solution of others that may be proposed to him, or that may be invented by his own ingenuity.

MISCELLANEOUS PROPOSITIONS.

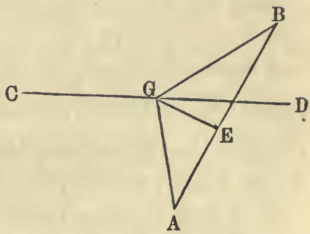
We shall conclude this book, and the subject of Geometry, by offering the following propositions, — some theorems, others problems, and some a combination of both, —not only for the purpose of impressing, by application, the geometrical principles which have now been established, but for the not less important purpose of cultivating the power of independent investigation.

After one or two propositions in which the beginner will be assisted in the analysis and construction, we shall leave him to his own resources, with the caution that a patient consideration of all the conditions in each case, and not mere trial operation, is the only process by which he can hope to reach the desired result.

1. From two given points, to draw two equal straight lines, which shall meet in the same point in a given straight line.

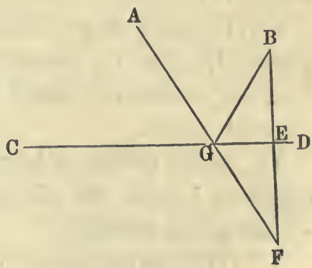


Let A and B be the given points, and CD the given straight line. Produce the perpendicular to the straight line AB at its middle point, until it meets CD in G. It is then easily proved that G is the point in CD in which the equal lines from A and B must meet. That is, that $AG = BG$.



If the points A and B were on opposite sides of CD, the directions for the construction would be the same, and we should have this figure; but the reasoning by which we prove $AG = BG$ would be unchanged.

2. From two given points on the same side of a given straight line, to draw two straight lines which shall meet in the given line, and make equal angles with it.



Let CD be the given line, and A and B the given points.

From B draw BE perpendicular to CD, and produce the perpendicular to F, making EF equal to BE; then draw AF, and from the point G, in which it intersects CD, draw GB. Now, $\angle BGE = \angle EGF = \angle AGC$. Hence, the angles BGD and AGC are equal, and the lines AG and BG meet in a common point in the line CD, and made equal angles with that line.

3. If, from a point without a circle, two straight lines be drawn to the concave part of the circumference, making equal angles with the line joining the same point and the center, the parts of these lines which are intercepted within the circle, are equal.

4. If a circle be described on the radius of another circle, any straight line drawn from the point where they meet, to the outer circumference, is bisected by the interior one.

5. From two given points on the same side of a line given in position, to draw two straight lines which shall contain a given angle, and be terminated in that line.

6. If, from any point without a circle, lines be drawn touching the circle, the angle contained by the tangents is double the angle contained by the line joining the points of contact and the diameter drawn through one of them.

7. If, from any two points in the circumference of a circle, there be drawn two straight lines to a point in a tangent to that circle, they will make the greatest angle when drawn to the point of contact.

8. From a given point within a given circle, to draw a straight line which shall make, with the circumference, an angle, less than any angle made by any other line drawn from that point.

9. If two circles cut each other, the greatest line that can be drawn through either point of intersection, is that which is parallel to the line joining their centers.

10. If, from any point within an equilateral triangle, perpendiculars be drawn to the sides, their sum is equal to a perpendicular drawn from any of the angles to the opposite side.

11. If the points of bisection of the sides of a given triangle be joined, the triangle so formed will be one fourth of the given triangle.

12. The difference of the angles at the base of any triangle, is double the angle contained by a line drawn from the vertex perpendicular to the base, and another bisecting the angle at the vertex.

13. If, from the three angles of a triangle, lines be drawn to the points of bisection of the opposite sides, these lines intersect each other in the same point.

14. The three straight lines which bisect the three angles of a triangle, meet in the same point.

15. The two triangles, formed by drawing straight lines from any point within a parallelogram to the extremities of two opposite sides, are, together, one half the parallelogram.

16. The figure formed by joining the points of bisection of the sides of a trapezium, is a parallelogram.

17. If squares be described on three sides of a right-angled triangle, and the extremities of the adjacent sides be joined, the triangles so formed are equivalent to the given triangle, and to each other.

18. If squares be described on the hypotenuse and sides of a right-angled triangle, and the extremities of the sides of the former, and the adjacent sides of the others, be joined, the sum of the squares of the lines joining them will be equal to five times the square of the hypotenuse.

19. The vertical angle of an oblique-angled triangle inscribed in a circle, is greater or less than a right angle, by the angle contained between the base and the diameter drawn from the extremity of the base.

20. If the base of any triangle be bisected by the diameter of its circumscribing circle, and, from the extremity of that diameter, a perpendicular be let fall upon the longer side, it will divide that side into segments, one of which will be equal to one half the sum, and the other to one half the difference, of the sides.

21. A straight line drawn from the vertex of an equilateral triangle inscribed in a circle, to any point in the opposite circumference, is equal to the sum of the two lines which are drawn from the extremities of the base to the same point.

22. The straight line bisecting any angle of a triangle

inscribed in a given circle, cuts the circumference in a point which is equi-distant from the extremities of the side opposite to the bisected angle, and from the center of a circle inscribed in the triangle.

23. If, from the center of a circle, a line be drawn to any point in the chord of an arc, the square of that line, together with the rectangle contained by the segments of the chord, will be equal to the square described on the radius.

24. If two points be taken in the diameter of a circle, equidistant from the center, the sum of the squares of the two lines drawn from these points to any point in the circumference, will be always the same.

25. If, on the diameter of a semicircle, two equal circles be described, and in the space included by the three circumferences, a circle be inscribed, its diameter will be $\frac{2}{3}$ the diameter of either of the equal circles.

26. If a perpendicular be drawn from the vertical angle of any triangle to the base, the difference of the squares of the sides is equal to the difference of the squares of the segments of the base.

27. The square described on the side of an equilateral triangle, is equal to three times the square of the radius of the circumscribing circle.

28. The sum of the sides of an isosceles triangle is less than the sum of the sides of any other triangle on the same base and between the same parallels.

29. In any triangle, given one angle, a side adjacent to the given angle, and the difference of the other two sides, to construct the triangle.

30. In any triangle, given the base, the sum of the other two sides, and the angle opposite the base, to construct the triangle.

31. In any triangle, given the base, the angle opposite to the base, and the difference of the other two sides, to construct the triangle.

BOOK IX.

SPHERICAL GEOMETRY.

DEFINITIONS.

1. **Spherical Geometry** has for its object the investigation of the properties, and of the relations to each other, of the portions of the surface of a sphere which are bounded by the arcs of its great circles.

2. A **Spherical Polygon** is a portion of the surface of a sphere bounded by three or more arcs of great circles, called the *sides* of the polygon.

3. The **Angles** of a spherical polygon are the angles formed by the bounding arcs, and are the same as the angles formed by the planes of these arcs.

4. A **Spherical Triangle** is a spherical polygon having but three sides, each of which is less than a semi-circumference.

5. A **Lune** is a portion of the surface of a sphere included between two great semi-circumferences having a common diameter.

6. A **Spherical Wedge**, or **Ungula**, is a portion of the solid sphere included between two great semi-circles having a common diameter.

7. A **Spherical Pyramid** is a portion of a sphere bounded by the faces of a solid angle having its vertex at the center, and the spherical polygon which these faces intercept on the surface. This spherical polygon is called the *base* of the pyramid.

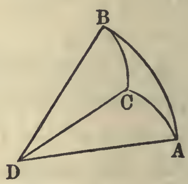
8. The **Axis** of a great circle of a sphere is that diameter of the sphere which is perpendicular to the plane of the circle. This diameter is also the axis of all small circles parallel to the great circle.

9. A **Pole** of a circle of a sphere is a point on the surface of the sphere equally distant from every point in the circumference of the circle.

10. **Supplemental, or Polar Triangles,** are two triangles on a sphere, so related that the vertices of the angles of either triangle are the poles of the sides of the other.

PROPOSITION I.

Any two sides of a spherical triangle are together greater than the third side.



Let AB, AC, and BC, be the three sides of the triangle, and D the center of the sphere.

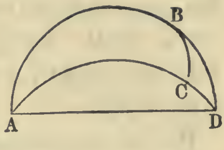
The angles of the planes that form the solid angle at D, are measured by the arcs AB, AC, and BC. But any two of these angles are together greater than the third angle, (Th. 18, B. VI). Therefore, any two sides of the triangle are, together, greater than the third side.

Hence the proposition.

PROPOSITION II.

The sum of the three sides of any spherical triangle is less than the circumference of a great circle.

Let ABC be a spherical triangle; the two sides, AB and AC, produced, will meet at the point which is diametrically opposite to A, and the arcs, ABD and ACD are together equal to a great circle. But, by the last proposition, BC is less than the two arcs, BD and DC. Therefore, $AB + BC + AC$, is less than $ABD + ACD$; that is, less than a great circle.

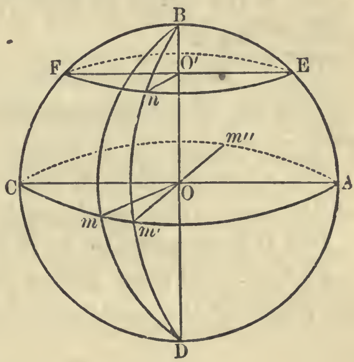


Hence the proposition.

PROPOSITION III.

The extremities of the axis of a great circle of a sphere are the poles of the great circle, and these points are also the poles of all small circles parallel to the great circle.

Let O be the center of the sphere, and BD the axis of the great circle, $Cm\ Am''$; then will B and D, the extremities of the axis, be the poles of the circle, and also the poles of any parallel small circle, as FnE.



For, since BD is perpendicular to the plane of the circle, $Cm\ Am''$, it is perpendicular to the lines OA, Om', Om'', etc., passing through its foot in the plane, (Def. 2, B. VI); hence, all the arcs, Bm, Bm', etc., are quadrants, as are also the arcs Dm, Dm', etc. The points B and D are, therefore, each equally distant from all the points in the circumference, $Cm\ Am''$; hence, (Def. 9), they are its poles.

Again, since the radius, OB, is perpendicular to the plane of the circle, $Cm\ Am''$, it is also perpendicular to the plane of the parallel small circle, FnE, and passes through its center, O'. Now, the chords of the arcs, BF, Bn, BE, etc., being oblique lines, meeting the plane of the small circle at equal distances from the foot of the perpendicular, BO', are all equal, (Th. 4, B. VI); hence, the arcs themselves are equal, and B is one pole of the circle, FnE. In like manner we prove the arcs, DF, Dn, DE, etc., equal, and therefore D is the other pole of the same circle.

Hence the proposition, etc.

Cor. 1. *A point on the surface of a sphere at the distance of a quadrant from two points in the arc of a great circle, not at the extremities of a diameter, is a pole of that arc.*

For, if the arcs, Bm, Bm', are each quadrants, the angles, BOm and BOm', are each right angles; and hence, BO is perpendicular to the plane of the lines, Om and Om', which is the plane of the arc, $m\ m'$; B is therefore the pole of this arc.

Cor. 2. *The angle included between the arc of a great circle and the arc of another great circle, connecting any of its points with the pole, is a right angle.*

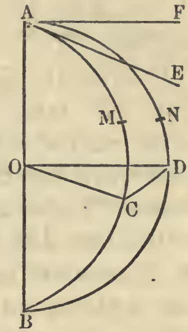
For, since the radius, BO, is perpendicular to the plane of the circle, $Cm\ Am''$, every plane passed through this radius is perpendicular to the plane of the circle; hence, the plane of the arc Bm is perpendicular to that of the arc Cm; and the angle of the arcs is that of their planes.

PROPOSITION IV.

The angle formed by two arcs of great circles which intersect each other, is equal to the angle included between the tangents to these arcs at their point of intersection, and is measured by that arc of a great circle whose pole is the vertex of the angle, and which is limited by the sides of the angle or the sides produced.

Let AM and AN be two arcs intersecting at the point A, and let AE and AF be the tangents to these arcs at this point. Take AC and AD, each quadrants, and draw the arc CD, of which A is the pole, and OC and OD are the radii.

Now, since the planes of the arcs intersect in the radius OA, and AE is a tangent to one arc, and AF a tangent to the other, at the common point A, these tangents form with each other an angle which is the measure of the angle of the planes of the arcs; but the angle of the planes of the arcs is taken as the angle included by the arcs, (Def. 3).



Again, because the arcs, AC and AD, are each quadrants, the angles, AOC, AOD, are right angles; hence the radii, OC and OD, which lie, one in one face, and the other in the other face, of the diedral angle formed by the planes of the arcs, are perpendicular to the common intersection of these faces at the same point. The angle, COD, is therefore the angle of the planes, and consequently the angle of the arcs; but the angle COD is measured by the arc CD.

Hence the proposition.

Cor. 1. Since the angles included between the arcs of great circles on a sphere, are measured by other arcs of great circles of the same sphere, we may compare such angles with each other, and construct angles equal to other angles, by processes which do not differ in principle from those by which plane angles are compared and constructed.

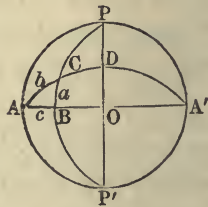
Cor. 2. Two arcs of great circles will form, by their intersection, four angles, the opposite or vertical ones of which will be equal, as in the case of the angles formed by the intersection of straight lines, (Th. 4, B. I).

PROPOSITION V.

The surface of a hemisphere may be divided into three right-angled and four quadrantal triangles, and one of these right-angled triangles will be so related to the other two, that two of its sides and one of its angles will be complemental to the *sides of one of them, and two of its sides supplemental to two of the sides of the other.*

Let ABC be a right-angled spherical triangle, right angled at B.

Produce the sides, AB and AC, and they will meet at A', the opposite point on the sphere. Produce BC, both ways, 90° from the point B, to P and P', which are, therefore, poles to the arc AB, (Prop. 3). Through A, P, and the center of the sphere, pass a plane, cutting the sphere into two equal parts, forming a great circle on the sphere, which great circle will be represented by the circle $PAP'A'$ in the figure. At right angles to this plane, pass another plane, cutting the sphere into two equal parts; this great circle is represented in the figure by the straight line, POP'. A and A' are the poles to the great circle, POP'; and P and P' are the poles to the great circle, ABA'.



Now, CPD is a spherical triangle, right-angled at D, and its sides CP and CD are complemental respectively to the sides BC and AC of the $\triangle\ ABC$, and its side PD is complemental to the arc DO, which measures the $\angle BAC$ of the same triangle. Again, the $\triangle\ A'BC$ is right-angled at B, and its sides $A'C$, $A'B$, are supplemental respectively to the sides AC, AB, of the $\triangle ABC$. Therefore, the three right-angled $\triangle$'s, ABC, CPD, and $A'BC$, have the required relations. In the $\triangle\ ACP$, the side AP is a quadrant, and for this reason the $\triangle$ is called a quadrantal triangle. So also, are the $\triangle$'s $A'CP$, ACP', and $P'CA'$, quadrantal triangles. Hence the proposition.

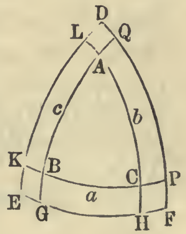
SCHOLIUM.—In every triangle there are *six* elements, three sides and three angles, called the parts of the triangle.

Now, if all the parts of the triangle ABC are known, the parts of each of the $\triangle$'s, PCD and $A'BC$, are as completely known. And when the parts of the $\triangle\ PCD$ are known, the parts of the $\triangle$'s ACP

and $A'CP$ are also known; for, the side PD measures each of the $\angle$'s PAC and $PA'C$, and the angle CPD, added to the right angle $A'PD$, gives the $\angle A'PC$, and the $\angle CPA$ is supplemental to this. Hence, the solution of the $\triangle ABC$ is a solution of the two right-angled and four quadrantal $\triangle$'s, which together with it make up the surface of the hemisphere.

PROPOSITION VI.

If there be three arcs of great circles whose poles are the angular points of a spherical triangle, such arcs, if produced, will form another triangle, whose sides will be supplemental to the angles of the first triangle, and the sides of the first triangle will be supplemental to the angles of the second.



Let the arcs of the three great circles be GH, PQ, KL, whose poles are respectively A, B, and C. Produce the three arcs until they meet in D, E, and F. We are now to prove that E is the pole of the arc AC; D the pole of the arc BC; F the pole to the arc AB. Also, that the side EF, is supplemental to the angle A; ED to the angle C; and DF to the angle B; and also, that the side AC is supplemental to the angle E, etc.

A pole is 90° from any point in the circumference of its great circle; and, therefore, as A is the pole of the arc GH, the point A is 90° from the point E. As C is the pole of the arc LK, C is 90° from any point in that arc; therefore, C is 90° from the point E; and E being 90° from both A and C, it is the pole of the arc AC. In the same manner, we may prove that D is the pole of BC, and F the pole of AB.

Because A is the pole of the arc GH, the arc GH measures the angle A, (Prop. 4); for a similar reason, PQ measures the angle B, and LK measures the angle C.

Because E is the pole of the arc AC, $\quad EH = 90°$

Or, $\quad EG + GH = 90°$

For a like reason, $\quad FH + GH = 90°$

Adding these two equations, and observing that $GH = A$, and afterward transposing one A, we have,

$$EG + GH + FH = 180° - A.$$

$$\left.\begin{array}{lr} \text{Or,} & EF = 180° - A \\ \text{In like manner,} & FD = 180° - B \\ \text{And,} & DE = 180° - C \end{array}\right\} \quad (a)$$

But the arc $(180° - A)$, is a supplemental arc to A, by the definition of arcs; therefore, the three sides of the triangle DEF, are supplements of the angles A, B, C, of the triangle ABC.

Again, as E is the pole of the arc AC, the whole angle E is measured by the whole arc LH.

$$\begin{array}{lr} \text{But,} & AC + CH = 90° \\ \text{Also,} & \underline{AC + AL = 90°} \end{array}$$

By addition, $AC + AC + CH + AL = 180°$

By transposition, $AC + CH + AL = 180° - AC$

$$\left.\begin{array}{lr} \text{That is,} & LH, \text{ or } E = 180° - AC \\ \text{In the same manner,} & F = 180° - AB \\ \text{And,} & D = 180° - BC \end{array}\right\} \quad (b)$$

That is, the sides of the first triangle are supplemental to the angles of the second triangle.

PROPOSITION VII.

The sum of the three angles of any spherical triangle, is greater than two right angles, and less than six right angles.

Add equations (a), of the last proposition. The first member of the equation so formed will be the sum of the three sides of a spherical triangle, which sum we may designate by S. The second member will be 6 right angles (there being 2 right angles in each 180°) less the three angles A, B, and C.

That is, $\qquad S = 6 \text{ right angles} - (A + B + C)$

By Prop. 2, the sum S is less than 4 right angles;

therefore, to it add s, a sufficient quantity to make 4 right angles. Then,

$$4 \text{ right angles} = 6 \text{ right angles} - (A + B + C) + s$$

Drop or cancel 4 right angles from both members, and transpose $(A + B + C)$.

Then, $$A + B + C = 2 \text{ right angles} + s.$$

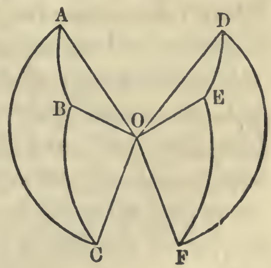
That is, the three angles of a spherical triangle make a greater sum than two right angles by the indefinite quantity s, which quantity is called the *spherical excess*, and is greater or less according to the size of the triangle.

Again, the sum of the angles is less than 6 right angles. There are but *three* angles in any triangle, and each one of them must be less than 180°, or 2 right angles. For, an angle is the inclination of two lines or two planes; and when two planes incline by 180°, the planes are parallel, or are in one and the same plane; therefore, as neither angle can be equal to 2 right angles, the three can never be equal to 6 right angles.

PROPOSITION VIII.

On the same sphere, or on equal spheres, triangles which are mutually equilateral are also mutually equiangular; and, conversely, triangles which are mutually equiangular are also mutually equilateral, equal sides lying opposite equal angles.

First.—Let ABC and DEF, in which $AB = DE$, $AC = DF$, and $BC = EF$, be two triangles on the sphere whose center is O; then will the $\angle A$, opposite the side BC, in the first triangle, be equal the $\angle D$, opposite the equal side EF, in the second; also $\angle B = \angle E$, and $\angle C = \angle F$.



For, drawing the radii to the vertices of the angles of these triangles, we may conceive O to be the common vertex of two triedral angles, one of which is bounded by the plane angles AOB, BOC, and AOC, and the other by the plane angles DOE, EOF, and DOF. But the plane angles bounding the one of these triedral angles, are equal to the plane angles bounding the other, each to each, since they are measured by the equal sides of the two triangles. The planes of the equal arcs in the two triangles are therefore equally inclined to each other, (Th. 20, B. VI); but the angles included between the planes of the arcs are equal to the angles formed by the arcs, (Def. 3).

Hence the $\angle A$, opposite the side BC, in the $\triangle ABC$, is equal to the $\angle D$, opposite the equal side EF, in the other triangle; and for a similar reason, the $\angle B = \angle E$, and the $\angle C = \angle F$.

Second.—If, in the triangles ABC and DEF, being on the same sphere whose center is O, the $\angle A = \angle D$, the $\angle B = \angle E$, and the $\angle C = \angle F$; then will the side AB, opposite the $\angle C$, in the first, be equal to the side DE, opposite the equal $\angle F$, in the second; and also the side AC equal to the side DF, and the side BC equal to the side EF.

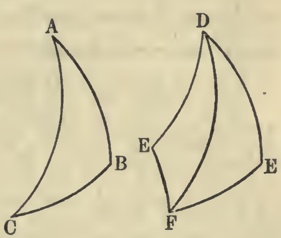
For, conceive two triangles, denoted by $A'B'C'$ and $D'E'F'$, supplemental to ABC and DEF, to be formed; then will these supplemental triangles be mutually equilateral, for their sides are measured by 180° less the opposite and equal angles of the triangles ABC and DEF, (Prop. 6); and being mutually equilateral, they are, as proved above, mutually equiangular. But the triangles ABC and DEF are supplemental to the triangles $A'B'C'$ and $D'E'F'$; and their sides are therefore measured severally by 180° less the opposite and equal angles of the triangles $A'B'C'$ and $D'E'F'$, (Prop. 6).

Hence the triangles ABC and DEF, which are mutually equiangular, are also mutually equilateral.

SCHOLIUM.—With the three arcs of great circles, AB, AC, and BC, either of the two triangles, ABC, DEF, may be formed; but it is evident that these two triangles cannot be made to coincide, though they are both mutually equilateral and mutually equiangular. Spherical triangles on the same sphere, or on equal spheres, in which the sides and angles of the one are equal to the sides and angles of the other, each to each, but are not themselves capable of superposition, are called *symmetrical triangles.*

PROPOSITION IX.

On the same sphere, or on equal spheres, triangles having two sides of the one equal to two sides of the other, each to each, and the included angles equal, have their remaining sides and angles equal.



Let ABC and DEF be two triangles, in which $AB = DE$, $AC = DF$, and the angle $A =$ the angle D; then will the side BC be equal to the side FE, the $\angle B =$ the $\angle E$, and $\angle C = \angle F$.

For, if DE lies on the same side of DF that AB does of AC, the two triangles, ABC and DEF, may be applied the one to the other, and they may be proved to coincide, as in the case of plane triangles. But, if DE does not lie on the same side of DF that AB does of AC, we may construct the triangle which is symmetrical with DEF; and this symmetrical triangle, when applied to the triangle ABC, will exactly coincide with it. But the triangle DEF, and the triangle symmetrical with it, are not only mutually equilateral, but also are mutually equiangular, the equal angles lying opposite the equal sides, (Prop. 8); and as the one or the other will coincide with the triangle ABC, it follows that the triangles, ABC and DEF, are either absolutely or symmetrically equal.

Cor. On the same sphere, or on equal spheres, triangles having two angles of the one equal to two angles of the other, each to each, and the included sides equal, have their remaining sides and angles equal.

For, if $\angle A = \angle D$, $\angle B = \angle E$, and side AB = side DE, the triangle DEF, or the triangle symmetrical with it, will exactly coincide with $\triangle ABC$, when applied to it as in the case of plane triangles; hence, the sides and angles of the one will be equal to the sides and angles of the other, each to each.

PROPOSITION X.

In an isosceles spherical triangle, the angles opposite the equal sides are equal.

Let ABC be an isosceles spherical triangle, in which AB and AC are the equal sides; then will $\angle B = \angle C$.

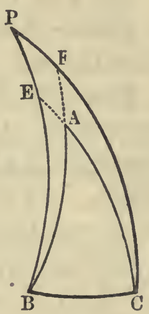
For, connect the vertex A with D, the middle point of the base, by the arc of a great circle, thus forming the two mutually equilateral triangles, ADB and ADC. They are mutually equilateral, because AD is common, $BD = DC$ by construction, and $AB = AC$ by supposition; hence they are mutually equiangular, the equal angles being opposite the equal sides, (Prop. 8). The angles B and C, being opposite the common side AD, are therefore equal.

Cor. The arc of a great circle which joins the vertex of an isosceles spherical triangle with the middle point of the base, is perpendicular to the base, and bisects the vertical angle of the triangle; and, conversely, the arc of a great circle which bisects the vertical angle of an isosceles spherical triangle, is perpendicular to, and bisects the base.

PROPOSITION XI.

If two angles of a spherical triangle are equal, the opposite sides are also equal, and the triangle is isosceles.

In the spherical triangle, ABC, let the $\angle B = \angle C$; then will the sides, AB and AC, opposite these equal angles, be equal.



For, let P be the pole of the base, BC, and draw the arcs of great circles, PB, PC; these arcs will be quadrants, and at right angles to BC, (Cor. 2, Prop. 3). Also, produce CA and BA to meet PB and PC, in the points E and F. Now, the angles, PBF and PCE, are equal, because the first is equal to 90° less the $\angle ABC$, and the second is equal to 90° less the equal $\angle ACB$; hence, the $\triangle$'s, PBF and PCE, are equal in all their parts, since they have the $\angle P$ common, the $\angle PBF = \angle PCE$, and the side PB equal to the side PC, (Cor., Prop. 9). PE is therefore equal to PF, and $\angle PEC = \angle PFB$.

Taking the equals PF and PE, from the equals PC and PB, we have the remainders, FC and EB, equal; and, from 180°, taking the $\angle$'s PFB and PEC, we have the remaining $\angle$'s, AFC and AEB, equal. Hence, the $\triangle$'s, AFC and AEB, have two angles of the one equal to two angles of the other, each to each, and the included sides equal; the remaining sides and angles are therefore equal, (Cor., Prop. 9). Therefore, AC is equal to BA, and the $\triangle$ ABC is isosceles.

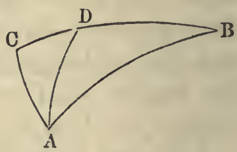
Cor. An equiangular spherical triangle is also equilateral, and the converse.

Remark.—In this demonstration, the pole of the base, BC, is supposed to fall without the triangle, ABC. The same figure may be used for the case in which the pole falls within the triangle; the modification the demonstration then requires is so slight and obvious, that it would be superfluous to suggest it.

PROPOSITION XII.

The greater of two sides of a spherical triangle is opposite the greater angle; and, conversely, the greater of two angles of a spherical triangle is opposite the greater side.

Let ABC be a spherical triangle, in which the angle A is greater than the angle B; then is the side BC greater than the side AC.



Through A draw the arc of a great circle, AD, making, with AB, the angle BAD equal to the angle ABD. The triangle, DAB, is isosceles, and $DA = DB$, (Prop. 11).

In the $\triangle\, ACD$, $CD + AD > AC$, (Prop. 1.); or, substituting for AD its equal DB, we have,

$$CD + DB > AC.$$

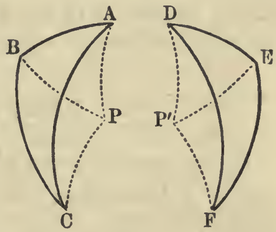
If in the above inequality we now substitute CB for $CD + DB$, it becomes $CB > CA$.

Conversely; if the side CB be greater than the side CA, then is the $\angle A >$ the $\angle B$. For, if the $\angle A$ is not greater than the $\angle B$, it is either equal to it, or less than it. The $\angle A$ is not equal to the $\angle B$; for if it were, the triangle would be isosceles, and CB would be equal to CA, which is contrary to the hypothesis. The $\angle A$ is not less than the $\angle B$; for if it were, the side CB would be less than the side CA, by the first part of the proposition, which is also contrary to the hypothesis; hence, the $\angle A$ must be greater than the $\angle B$.

PROPOSITION XIII.

Two symmetrical spherical triangles are equal in area

Let ABC and DEF be two $\triangle$'s on the same sphere, having the sides and angles of the one equal to the sides and angles of the other, each to each, the triangles themselves not admitting of superposition. It is to be proved that these $\triangle$'s have equal areas.



Let P be the pole of a small circle passing through the three points, ABC, and connect P with each of the points, A, B, and C, by arcs of great circles. Next, through E draw the arc of a great circle, EP', making the angle DEP' equal to the angle ABP. Take $EP' = BP$, and draw the arcs of great circles, $P'D$, $P'F$.

The $\triangle$'s, ABP and DEP', are equal in all their parts, because $AB = DE$, $BP = EP'$, and the $\angle ABP = \angle DEP'$, (Prop. 9). Taking from the $\angle ABC$ the $\angle ABP$, and from the $\angle DEF$ the $\angle DEP'$, we have the remaining angles, PBC and $P'EF$, equal; and therefore the $\triangle$'s, BCP and EFP', are also equal in all their parts.

Now, since the $\triangle$'s, ABP and DEP', are isosceles, they will coincide when applied, as will also the $\triangle$'s, BCP and EFP', for the same reason. The polygonal areas, $ABCP$ and $DEFP'$, are therefore equivalent. If from the first we take the isosceles triangle, PAC, and from the second the equal isosceles triangle, $P'DF$, the remainders, or the triangles ABC and DEF, will be equivalent.

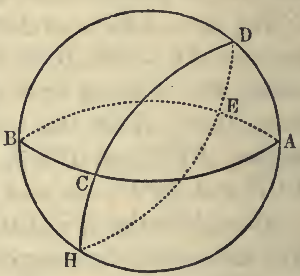
REMARK.—It is assumed in this demonstration that the pole P falls without the triangle. Were it to fall within, instead of without, no other change in the above process would be required than to add the isosceles triangles, PAC, $P'DF$, to the polygonal areas, to get the areas of the triangles, ABC, DEF.

Cor. Two spherical triangles on the same sphere, or on equal spheres, will be equivalent — 1st, when they are mutually equilateral; — 2d, when they are mutually equiangular; — 3d, when two sides of the one are equal to two sides of the other, each to each, and the included angles are equal; — 4th, when two angles of the one are equal to two angles of the other, each to each, and the included sides are equal.

PROPOSITION XIV.

If two arcs of great circles intersect each other on the surface of a hemisphere, the sum of either two of the opposite triangles thus formed will be equivalent to a lune whose angle is the corresponding angle formed by the arcs.

Let the great circle, $AEBC$, be the base of a hemisphere, on the surface of which the great semi-circumferences, BDA and CDE, intersect each other at D; then will the sum of the opposite triangles, BDC and DAE, be equivalent to the lune whose angle is BDC; and the sum of the opposite triangles, CDA and BDE, will be equivalent to the lune whose angle is CDA.



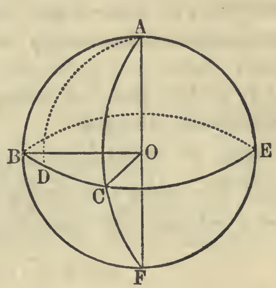
Produce the arcs, BDA and CDE, until they intersect on the opposite hemisphere at H; then, since CDE and DEH are both semi-circumferences of a great circle, they are equal. Taking from each the common part DE, we have $CD = HE$. In the same way we prove $BD = HA$, and $AE = BC$. The two triangles, BDC and HAE, are therefore mutually equilateral, and hence they are equivalent, (Prop. 13). But the two triangles, HAE and ADE, together, make up the lune

DEHAD; hence the sum of the △'s, *BDC* and *ADE*, is equivalent to the same lune.

By the same course of reasoning, we prove that the sum of the opposite △'s, *DAC* and *DBE*, is equivalent to the lune *DCHAD*, whose angle is *ADC*.

PROPOSITION XV.

The surface of a lune is to the whole surface of the sphere, as the angle of the lune is to four right angles; or, as the arc which measures that angle is to the circumference of a great circle.



Let *ABFCA* be a lune on the surface of a sphere, and *BCE* an arc of a great circle, whose poles are *A* and *F*, the vertices of the angles of the lune. The arc, *BC*, will then measure the angles of the lune. Take any arc, as *BD*, that will be contained an exact number of times in *BC*, and in the whole circumference, *BCEB*, and, beginning at *B*, divide the arc and the circumference into parts equal to *BD*, and join the points of division and the poles, by arcs of great circles. We shall thus divide the whole surface of the sphere into a number of equal lunes. Now, if the arc *BC* contains the arc *BD* m times, and the whole circumference contains this arc n times, the surface of the lune will contain m of these partial lunes, and the surface of the sphere will contain n of the same; and we shall have,

Surf. lune : surf. sphere :: $m : n$.

But, $m : n$ :: *BC* : circumference great circle;

hence, surf. lune : surf. sphere :: *BC* : cir. great circle;

or, surf. lune : surf. sphere :: $\angle BOC$: 4 right angles.

This demonstration assumes that BD is a common measure of the arc, BC, and the whole circumference. It may happen that no finite common measure can be found; but our reasoning would remain the same, even though this common measure were to become indefinitely small.

Hence the proposition.

Cor. 1. Any two lunes on the same sphere, or on equal spheres, are to each other as their respective angles.

SCHOLIUM. — Spherical triangles, formed by joining the pole of an arc of a great circle with the extremities of this arc by the arcs of great circles, are isosceles, and contain two right angles. For this reason they are called *bi-rectangular*. If the base is also a quadrant, the vertex of either angle becomes the pole of the opposite side, and each angle is measured by its opposite side. The three angles are then right angles, and the triangle is for this reason called *tri-rectangular*. It is evident that the surface of a sphere contains eight of its tri-rectangular triangles.

Cor. 2. Taking the right angle as the unit of angles, and denoting the angle of a lune by A, and the surface of a tri-rectangular triangle by T, we have,

$$\text{surf. of lune} : 8T :: A : 4;$$

whence, $$\text{surf. of lune} = 2A \times T.$$

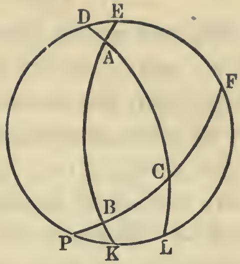
Cor. 3. A spherical ungula bears the same relation to the entire sphere, that the lune, which is the base of the ungula, bears to the surface of the sphere; and hence, any two spherical ungulas in the same sphere, or in equal spheres, are to each other as the angles of their respective lunes.

PROPOSITION XVI.

The area of a spherical triangle is measured by the excess of the sum of its angles over two right angles, multiplied by the tri-rectangular triangle.

Let ABC be a spherical triangle, and $DEFLK$ the circumference of the base of the hemisphere on which this triangle is situated.

Produce the sides of the triangle until they meet this circumference in the points, D, E, F, L, K, and P, thus forming the sets of opposite triangles, DAE, AKL; BEF, BPK; CFL, CDP.



Now, the triangles of each of these sets are together equal to a lune, whose angle is the corresponding angle of the triangle, (Prop. 14); hence we have,

$$\triangle DAE + \triangle AKL = 2A \times T, \text{(Prop. 15, Cor. 2).}$$
$$\triangle BEF + \triangle BPK = 2B \times T.$$
$$\triangle CFL + \triangle CDP = 2C \times T.$$

If the first members of these equations be added, it is evident that their sum will exceed the surface of the hemisphere by twice the triangle ABC; hence, adding these equations member to member, and substituting for the first member of the result its value, $4T + 2\triangle ABC$, we have

$$4T + 2\triangle ABC = 2A.T + 2B.T + 2C.T$$

or, $\quad 2T + \triangle ABC = A.T + B.T + C.T$

whence, $\quad \triangle ABC = A.T + B.T + C.T - 2T.$

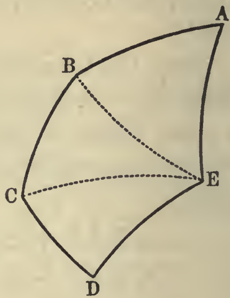
That is, $\quad \triangle ABC = (A + B + C - 2)\, T.$

But $A + B + C - 2$ is the excess of the sum of the angles of the triangle over two right angles, and T denotes the area of a tri-rectangular triangle.

Hence the proposition; *the area, etc.*

PROPOSITION XVII.

The area of any spherical polygon is measured by the excess of the sum of all its angles over two right angles, taken as many times, less two, as the polygon has sides, multiplied by the tri-rectangular triangle.



Let $ABCDE$ be a spherical polygon; then will its area be measured by the excess of the sum of the angles, A, B, C, D, and E, over two right angles taken a number of times which is two less than the number of sides, multiplied by T, the tri-rectangular triangle. Through the vertex of any of the angles, as E, and the vertices of the opposite angles, pass arcs of great circles, thus dividing the polygon into as many triangles, less two, as the polygon has sides. The sum of the angles of the several triangles will be equal to the sum of the angles of the polygon.

Now, the area of each triangle is measured by the excess of the sum of its angles over two right angles, multiplied by the tri-rectangular triangle. Hence the sum of the areas of all the triangles, or the area of the polygon, is measured by the excess of the sum of all the angles of the triangles over two right angles, taken as many times as there are triangles, multiplied by the tri-rectangular triangle. But there are as many triangles as the polygon has sides, less two.

Hence the proposition; *the area of any spherical polygon, etc.*

Cor. If S denote the sum of the angles of any spherical polygon, n the number of sides, and T the tri-rectangular triangle, the right angle being the unit of angles; the area of the polygon will be expressed by

$$[S - 2(n - 2)] \times T = (S - 2n + 4)\, T.$$